# SHY AND FIXED-DISTANCE COUPLINGS OF BROWNIAN MOTIONS ON MANIFOLDS

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ABSTRACT. In this paper we introduce three Markovian couplings of Brownian motions on smooth Riemannian manifolds without boundary which sit at the crossroad of two concepts. The first concept is the one of shy coupling put forward in [3] and the second concept is the lower bound on the Ricci curvature and the connection with couplings made in [28].

The first construction is the shy coupling, the second one is a fixed-distance coupling and the third is a coupling in which the distance between the processes is a deterministic exponentially function of time.

The simplest nontrivial manifold is the 2-dimensional sphere in $\mathbb{R}^3$, and in this case we give the explicit construction of all three types of couplings mentioned above and at first we use an extrinsic approach. Next, we construct part of these couplings on manifolds of constant curvature, this time using the intrinsic geometry.

Then we prove a full result which shows that an arbitrary Riemannian manifold satisfying some technical conditions supports shy couplings. Moreover, if in addition the Ricci curvature is non-negative, there exist fixed-distance couplings. Furthermore, if the Ricci curvature is bounded below by a positive constant, then there exists a coupling of Brownian motions for which the distance between the processes is deterministic and exponentially decaying. The constructions use the intrinsic geometry, and relies on an extension of the notion of frames which plays an important role for even dimensional manifolds.

As an application of the fixed-distance coupling we derive a maximum principle for the gradient of harmonic functions on manifolds with non-negative Ricci curvature.

## 1. INTRODUCTION

A first motivation of the present work is the following (stochastic) modification of the classical *Lion and Man problem* of Rado ([21]) on manifolds. Consider a Brownian Lion $X_t$ and a Brownian Man $Y_t$ running on a $d$-dimensional Riemannian manifold $M$ (for instance the unit sphere in $\mathbb{R}^3$).

We consider the following two versions of the classical Lion and Man problem.

**Problem 1** (Fast/Finite Time Coupling). *Can the Lion capture the Man?*

*More precisely, given two distinct starting points $x, y \in M$ and a Brownian motion $Y_t$ on $M$ starting at $y$, can one find a Brownian motion $X_t$ on $M$ starting at $x$ such that $\tau = \inf\{t \geq 0 : X_t = Y_t\}$ is almost surely finite (or almost surely bounded)? A weaker version of this problem is whether for a given $\epsilon > 0$ and a given Brownian motion $Y_t$ on $M$ starting at $y$ one can find a Brownian motion $X_t$ on $M$ starting at $x$ such that $\tau = \inf\{t \geq 0 : d(X_t, Y_t) = \epsilon\}$ is almost surely finite (or almost surely bounded). Here $d(x, y)$ stands for the Riemannian distance on $M$.*

**Problem 2** (Strong Shy Coupling). *Can the Man avoid being eaten by the Lion indefinitely?*

*More precisely, given two distinct starting points $x, y \in M$ and a Brownian motion $X_t$ on $M$ starting at $x$, can one find a Brownian motion $Y_t$ on $M$ starting at $y$ such that almost surely $X_t \neq Y_t$ for all $t \geq 0$? A stronger version of the question is whether the Brownian motion $Y_t$ can be chosen in such a way that there exists an $\epsilon > 0$ such that almost surely $d(X_t, Y_t) \geq \varepsilon$ for all $t \geq 0$.*

The notion of *shy coupling* of Brownian motions was introduced in [3] and subsequently studied in [4] and [16] and is a coupling for which, with positive probability, the distance between the two processes stays positive for all times. A stronger version of shyness ($\epsilon$-shyness, $\epsilon > 0$) asserts that with positive probability the distance between the processes is greater than $\epsilon$. In this paper we use this latter version of

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shyness, in the stronger sense where the distance between the processes is greater than $\epsilon$ with probability 1.

To set up the terminology, we mention that all couplings in the present paper are Markovian couplings in the sense of [3] and introduced in Section 2.

We note that on the unit sphere $S^2$, there is an immediate affirmative answer to Problem 1: one can define $X_t$ as the symmetric of $Y_t$ with respect to the plane of symmetry of $x$ and $y$. Since the Brownian motion $Y_t$ hits this plane in finite time, $\tau$ is finite almost surely, so the Lion is sure to capture the Man in finite time.

The above mentioned coupling is known in the literature as the *mirror coupling*, and it was introduced by Lindvall and Rogers [20] for processes defined on Euclidean spaces, and by Cranston in [9] and Kendall [15] in the case of processes defined on manifolds, the so-called *Cranston-Kendall mirror coupling*. It turns out that this coupling is a very useful and versatile construction when it comes to various geometric and analytic properties on manifolds. For instance, it was shown in [15], for the case of manifolds with Ricci curvature bounded uniformly from below by a positive constant, that the Man and the Lion must meet in finite time under this mirror coupling.

Geometrically, the mirror coupling makes the motions $X_t, Y_t$ move toward each other in the geodesic direction. Closely related coupling is the *synchronous* coupling in which the Brownian motions $X_t, Y_t$ move parallel to each other in the geodesic direction and was used for example in [2]. On a different note, continuous versions of couplings of Brownian motions are constructed in [1] and [25].

A synthetic notion of a lower bound on the Ricci curvature was settled in [22, 26, 27] and is a very useful tool in analysis on measure metric spaces. On the other hand, the notion of couplings and lower bound on Ricci curvature was pioneered in [23] and is particularly good for defining lower bounds on Ricci curvature in discrete spaces as it is for instance pointed out in [8, 19].

In this spirit, a second motivation of our work comes from [28, Corollary 1.4] which states the following.

**Corollary 3.** *On a complete Riemannian manifold $M$ the Ricci tensor satisfies $Ric \geq k$ if and only if there exits a conservative Markov process $(\Omega, \mathcal{A}, \mathbb{P}^z, Z_t)_{z\in M\times M, t\geq 0}$ with values in $M \times M$ such that the coordinate processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are Brownian motions on $M$ and such that for all $z = (x, y)$ and all $t \geq 0$,*

$$d(X_t, Y_t) \leq e^{-kt/2} d(x, y), \quad \mathbb{P}^z - a.s. \tag{1.1}$$

The coupling that is used in [28] under the hypothesis that $Ric \geq k$ is the synchronous coupling alluded above.

A natural question, and one of our interests in the present paper, is to see if one can find couplings of Brownian motions $X_t, Y_t$ such that (1.1) is saturated. For instance, if $k = 0$ this amounts to finding a fixed-distance coupling which is in fact a strong version of a shy coupling.

Here is an outline of the paper. Section 2 is about notations and basic results and notions. Then, Section 3, as a warm up, is about the existence of fixed-distance couplings on $\mathbb{R}^n$. Here we show that the only fixed-distance coupling is the trivial one, namely the translation coupling. In fact, we show a little more, namely that there is no distance-decreasing coupling in $\mathbb{R}^n$. This is to be contrasted to the fact that on $S^2$ it is possible to construct distance-decreasing couplings.

In Section 4 we prove the existence of the fast approaching, the repulsive, and the fixed-distance couplings on $S^2$, the two dimensional sphere. The construction is carried out using two ingredients: Stroock's representation of spherical Brownian motion and Kendall's characterization of co-adapted couplings of Brownian motions in Euclidean spaces (see [16]). From a differential geometric perspective this construction is extrinsic, in the sense that the sphere $S^2$ is seen as a submanifold of $\mathbb{R}^3$, and we take advantage of this in order to reduce the problem at hand to that of finding unitary matrices in $\mathbb{R}^3$ satisfying certain conditions. The intriguing part about this construction is that the same argument does not extend to higher dimensional spheres.

The above results raise the natural question whether this is the end of the story. Are the couplings obtained in the particular case of the sphere (or the Euclidean space) something accidental? What is really responsible for the existence of these couplings? Phrased differently, is it important that the sphere has so many symmetries or is it not? We answer this question from a differential geometric perspective, by proving that the existence of these couplings is in fact related to the intrinsic geometry of the sphere rather than the extrinsic one. Though this is not our main theme here, it is put forward in [17, 18] that in the context of maximality of couplings of Brownian motions the symmetries play an essential role.

We address the above questions in Section 5 in the case of manifolds of constant curvature (in any dimension), and treat the case of distance increasing/decreasing couplings. We switch from the extrinsic approach used in Section 4 to the intrinsic approach, and we prove that manifolds of constant negative curvature support shy couplings. In the particular case of the hyperbolic space this shows that we can construct couplings of Brownian motions which get away from each other exponentially fast (nothing very surprising after all). Interestingly enough, on manifolds of constant positive curvature we can construct shy couplings as well as couplings in which the processes get exponentially close to each other, but do not touch each other. In the particular case of spheres (in any dimension) we construct the exponentially decreasing/increasing couplings, which in the case of $S^2$, coincide with the ones from Section 4. We could have included here the construction of fixed-distance couplings on manifolds of constant curvature, but technically this is very much what we do later on, so we postponed this for Section 6. The reason for including these calculations on constant curvature manifolds is that on one hand this is an intrinsic approach and on the other hand the computation for the distance functions is instructive, relatively standard and simple.

In Section 6 we get a very general result. This states that on a complete $d$-dimensional Riemannian manifold $M$ with positive injectivity radius, the Ricci curvature uniformly bounded from below and the sectional curvature uniformly bounded from above we can construct shy couplings. This existence result of shy coupling on manifolds is also stated in Kendall [16] without proof but with a hint on how to do it. Our approach is a bit different. Moreover, if the Ricci curvature is in addition non-negative, we can also construct fixed-distance couplings. Finally, we show that if the Ricci curvature is actually bounded from below by a positive constant, then we can find fast approaching couplings, for which the distance between processes decays exponentially fast to $0$.

We want to point a few details about the techniques. In the first place we treat separately the cases when $d$ is odd, respectively even. In the case of odd dimensional manifolds we can carry out the proof based on the orthonormal frame bundle. For even dimensional manifolds we introduce the notion of $N$-frames at a point $x \in M$ which is an embedding of the tangent space $T_xM$ into $\mathbb{R}^N$. As it turns out, it suffices to use this construction for the particular case $N = d + 1$, however, for the general $N$ this may be of independent interest by itself. This is somewhat reminiscent of works on stochastic flows given for example in [10, 11].

Here is a brief exposition of the idea. Suppose we have $X_t$ a Brownian motions and want to exhibit another one $Y_t$ which is driven in some sense by $X_t$. From a loose point of view what we do first is to split the orthogonal to the tangent space at $X_t$ into orthogonal planes. This splitting is possible only if the dimension $d$ is odd. If this is the case, using the parallel transport along the geodesic, we can transport these planes at $X_t$ into orthogonal planes at $Y_t$. Next we want the components of driving Euclidean Brownian motion at $X_t$ in these planes to be transported at $Y_t$ using parallel transport along the geodesic joining $X_t$ and $Y_t$ and then rotated by the same angle (chosen appropriately) in each of the transported planes at $Y_t$. This is how we construct all three couplings first locally and then by patching them together to a global one. In the even dimensional case using the $d + 1$-frames we essentially add one more dimension to the tangent space and carry out the same program.

In Section 7 we discuss some geometric aspects related to the main result in the previous section (Theorem 12), and we present a localized version of the shy coupling, which is used in Section 8 in order to give some applications of the fixed-distance coupling to the maximum principle of norms of the gradient of harmonic functions and solutions to heat equations on manifolds with non-negative Ricci curvature. We end this section with a coming back to the motivations of the paper.

## 2. Preliminaries

We identify the vectors in $\mathbb{R}^3$ with the corresponding $3 \times 1$ column matrices, and for a vector $x \in \mathbb{R}^3$ we denote by $x'$ the transpose of $x$. The dot product of two vectors $x, y \in \mathbb{R}^3$ can be written in terms of matrix multiplication as $x \cdot y = x'y$. The Euclidian length of a vector $x \in \mathbb{R}^3$ is $\|x\| = \sqrt{x'x}$.

We denote by $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ the unit sphere in $\mathbb{R}^3$, and for $x, y \in S^2$ we let $d(x, y)$ be the length of the geodesic joining $x$ and $y$ on $S^2$ (the length of the smaller of the two arcs of a great circle containing $x$ and $y$, that is $d(x, y) = \arcsin \sqrt{1 - (x'y)^2} = 2 \arcsin \left(\frac{1}{2}\|x - y\|\right)$).

There are various ways of describing the spherical Brownian motion on $S^2$, that is the Brownian motion on $S^2$ (see for example [5]). In what follows we exploit the Stroock's representation of spherical Brownian motion ([24]), as the solution $X_t$ of the Itô stochastic differential equation

$$X_t = X_0 + \int_0^t \left(I - X_s X_s'\right) dB_s - \int_0^t X_s ds, \tag{2.1}$$

where $B_t$ is a 3-dimensional Brownian motion. The last term above may be thought as the pull needed in order to keep $X_t$ on the surface of $S^2$.

Given two non-parallel vectors $x, y \in S^2$ (i.e. $y \neq \pm x$), we denote by $R_{x,y}$ the $3 \times 3$ rotation matrix with axis $u = x \times y$ (the cross product of the vectors $x$ and $y$) and angle $\theta \in (0, \pi)$ equal to the angle between the vectors $x$ and $y$, so in particular $R_{x,y} u = u$ and $R_{x,y} x = y$. It is known that $R_{x,y}$ is an orthogonal matrix ($R^{-1} = R'$) and the following (Rodrigues' rotation) formula holds

$$R_{x,y} = \cos\theta I + [u]_\times + \frac{1}{1 + \cos\theta} u \otimes u, \tag{2.2}$$

where $[u]_\times = yx' - xy'$ is the cross-product matrix of $u = x \times y$, $\otimes$ denotes the tensor product ($u \otimes u = uu'$) and $I$ denotes the $3 \times 3$ identity matrix. Note that the above formula differs slightly from the usual one, due to the fact that we do not require the axis $u$ to be a unit vector.

When $y = \pm x$, the cross product $u = x \times y$ is the zero vector, so the rotation matrix $R_{x,\pm x}$ is not well defined in this case. However, if we define $R_{x,\pm x} = \pm I$ we see that the $R_{x,\pm x}$ is still a unitary matrix and satisfies $R_{x,\pm x} x = \pm x$. Moreover, taking the limit as $\theta \to 0$ (or $\pi$, depending on whether $y = x$ or $y = -x$), we see that (2.2) still holds.

By $M$ we denote Riemannian manifold. In this paper all Riemannian manifolds are assumed to be complete. For a given $d$-dimensional Riemannian manifold $M$, we use the standard notations from [12] or [25] to denote by $\mathcal{O}(M)$ the orthonormal frame bundle. For a given orthonormal frame $U$ at a point $x \in M$ and $\xi \in \mathbb{R}^d$, $H_\xi(U)$ is the horizontal lift of $U\xi \in T_x M$ at the point $U \in \mathcal{O}(M)$. We will use the simpler notation of $H_i$ for $H_{e_i}$, with $\{e_i\}_{i=1,\ldots,n}$ denoting the standard basis of $\mathbb{R}^d$.

We collect here some notions from differential geometry which will be used in the sequel. The reader is referred to [6] or [7] for basic notions and results. The curvature tensor $R_x$ at $x$ is $R_x(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ and the Ricci tensor is the contraction $Ric_x(X, Y) = \sum_{i=1}^d \langle R_x(X, E_i) E_i, Y \rangle$, where $\{E_i\}_{i=1,\ldots,d}$ is any orthonormal basis at $x$ and $X, Y \in T_x M$. This definition of the Ricci tensor does not depend on the choice of orthonormal basis, and in the particular case of surfaces it simplifies to $Ric_x(X, Y) = K_x \langle X, Y \rangle$, where $K$ is the Gauss curvature.

We denote by $d(x, y)$ the Riemannian distance between $x$ and $y$.

A geodesic on $M$ is a smooth curve $\gamma : [a, b] \to M$ such that $\ddot{\gamma}(s) = 0$ for each $s \in [a, b]$, where the dot represents the covariant derivative along $\gamma$. Throughout the paper we assume that the geodesics are running at unit speed. For a point $x \in M$, we define $C_x$ to be the cutlocus of $x$, that is the set of points $y \in M$ for which there is more then one minimizing geodesic between $x$ and $y$. We will also use the notation $Cut \subset M \times M$, defined as the set of all points $(x, y)$ which are at each other's cut-locus. For points $x, y \in M$ which are not at each other's cut-locus, we define $\gamma_{x,y}$ to be the unique unit speed minimizing curve joining $x$ and $y$.

The injectivity radius is the smallest number $i(M)$ such that any point $x \in M$, the exponential map at $x$ is a diffeomorphism on the ball of radius $i(M)$ in the tangent space $T_x M$.

Given a geodesic $\gamma$, a Jacobi field along $\gamma$ is a vector field $J(s)$ such that

$$\ddot{J}(s) + R_{\gamma(s)}(J(s), \dot{\gamma}(s))\dot{\gamma}(s) = 0, \tag{2.3}$$

where the dot represents the derivative along $\gamma$.

Given a vector field $V$ along a geodesic $\gamma$ defined on $[a,b]$, the index form $\mathcal{I}$ associated to it is defined as

$$\mathcal{I}(V,V)=\int_a^b (|\dot V(s)|^2-\langle R_{\gamma(s)}(V(s),\dot\gamma(s))\dot\gamma(s),V(s)\rangle)ds, \tag{2.4}$$

and using polarization $\mathcal{I}$ can be extended to a bilinear form on the space of vector fields along the geodesic $\gamma$. In the particular case when $J$ is a Jacobi field, an integration by parts formula shows that

$$\mathcal{I}(J,J)=\langle \dot J(b),J(b)\rangle-\langle \dot J(a),J(a)\rangle \tag{2.5}$$

where $[a,b]$ is the definition interval of $\gamma$.

A manifold has constant curvature $r$ if the sectional curvature is $r$ for all choices of the two dimensional plane, that is $\langle R_x(X,Y)Y,X\rangle=r$ for any $x\in M$ and any ortogonal unit vectors $X,Y\in T_xM$. In this case the Ricci curvature simplifies as well as the Jacobi field equation (2.3). We record here the calculation, as it will be used later on. Assume that $\gamma_{x,y}$ is the minimal geodesic between the points $x,y\in M$ which are not at each other's cut-locus, $\rho=d(x,y)$ and let $\xi\in T_xM$ and $\eta\in T_yM$ be two unit vectors. Consider $\xi(s)$ the extension of $\xi$ by parallel transport along $\gamma$ from $x$ to $y$, and similarly let $\eta(s)$ be the extension of $\eta$ by parallel transport from $y$ to $x$. The Jacobi field $J_{\xi,\eta}$ whose value at $x$ is $\xi$ and $\eta$ at $y$ can be computed as follows

$$J_{\xi,\eta}(s)=w_1(s)\xi(s)+w_2(s)\eta(s) \tag{2.6}$$

where $w_1$, $w_2$ solve the boundary value problems

$$\begin{cases}\ddot w_1+rw_1=0\\ w_1(0)=1\\ w_1(\rho)=0\end{cases}\quad\text{and}\quad\begin{cases}\ddot w_2+rw_2=0\\ w_2(0)=0\\ w_2(\rho)=1\end{cases},$$

whose solutions are

$$w_1(s)=\begin{cases}\frac{\sin(\sqrt{r}(\rho-s))}{\sin(\sqrt{r}\rho)}, & r\neq 0\\ \frac{\rho-s}{\rho}, & r=0\end{cases}\quad\text{and}\quad w_2(s)=\begin{cases}\frac{\sin(\sqrt{r}s)}{\sin(\sqrt{r}\rho)}, & r\neq 0\\ \frac{s}{\rho}, & r=0\end{cases}. \tag{2.7}$$

Next, we introduce the main notions regarding couplings. Recall that in general by a coupling we understand a pair of processes $(X_t,Y_t)$ defined on the same probability space, which are separately Markov, that is

$$P\left(X_{s+t}\in A|\,X_s=z,X_u:0\le u\le s\right)=P^z\left(X_t\in A\right)$$
$$P\left(Y_{s+t}\in A|\,Y_s=z,Y_u:0\le u\le s\right)=P^z\left(Y_t\in A\right)$$

for any measurable set $A$ in the state space of the processes.

The notion of *Markovian coupling* as used in [3] requires that in addition to the above, the joint process $(X_t,Y_t)$ is Markov and

$$\begin{aligned}P\left(X_{s+t}\in A|\,X_s=z,X_u,Y_u:0\le u\le s\right)=P^z\left(X_t\in A\right)\\ P\left(Y_{s+t}\in A|\,Y_s=z,X_u,Y_u:0\le u\le s\right)=P^z\left(Y_t\in A\right)\end{aligned} \tag{2.8}$$

for any measurable set $A$ in the state space of the processes.

The notion of *co-adapted coupling* (introduced by Kendall, [16]) is the same as the above but without the Markov property of $(X_t,Y_t)$.

The Markovian couplings are easily obtained for instance in the case when the process $(X_t,Y_t)$ is actually a diffusion on the manifold. This would be the ideal case, but we still get a Markovian coupling if we patch together diffusion process in a nice way. For example this will be the case of the main construction on manifolds, where we start the coupling following a diffusion up to a certain stopping time, then, from the point it stopped we run it independently according to another diffusion and then stop this at another stopping time and so on. We do this quietly without further details.

## 3. Distance-decreasing couplings in $\mathbb{R}^d$

In this section we first examine the distance-decreasing couplings in the Euclidean space $\mathbb{R}^d$. To be precise, we want to find all possible co-adapted couplings $(X_t, Y_t)$ of $d$ -dimensional Brownian motions, for which the distance $\|X_t - Y_t\|$ is a (deterministic) non-increasing function of $t \geq 0$.

By a result on co-adapted couplings (Lemma 6 in [16]), a co-adapted coupling $(X_t, Y_t)$ of Brownian motions in $\mathbb{R}^d$ can be represented as

$$Y_t = Y_0 + \int_0^t J_t dX_t + \int_0^t K_t dZ_t,$$

where $Z$ is a $d$-dimensional Brownian motion independent of $X$ (on a possibly larger filtration), and $J, K \in \mathcal{M}_{d\times d}$ are matrix-valued predictable random processes, satisfying the identity

$$J_t J_t' + K_t K_t' = I, \tag{3.1}$$

with $I$ denoting the $d \times d$ identity matrix.

Setting $W_t = X_t - Y_t$ and using Itô's formula we obtain

$$d\|W_t\|^2 = 2W_t' dW_t + \sum_{i=1}^d d\langle W^i\rangle_t = 2\left(X_t - Y_t\right)'\left(I - J_t\right) dX_t - 2\left(X_t - Y_t\right)' K_t dZ_t + \sum_{i=1}^d d\langle W^i\rangle_t.$$

Using the independence of $X$ and $Z$, and the relation 3.1 we obtain

$$\begin{aligned}
\sum_{i=1}^d d\langle W^i\rangle_t &= \operatorname{tr}\left(\left(I - J_t\right)'\left(I - J_t\right) + K_t' K_t\right) dt \\
&= \operatorname{tr}\left(I - J_t - J_t' + J_t' J_t + K_t' K_t\right) dt \\
&= 2\left(\operatorname{tr}\left(I\right) - \operatorname{tr}\left(J_t\right)\right) dt \\
&= 2\left(d - \operatorname{tr}\left(J_t\right)\right) dt.
\end{aligned}$$

From the last two equations we arrive at

$$d\|W_t\|^2 = 2\left(X_t - Y_t\right)'\left(I - J_t\right) dX_t - 2\left(X_t - Y_t\right)' K_t dZ_t + 2\left(d - \operatorname{tr}\left(J_t\right)\right) dt,$$

so the differential of the quadratic variation of the martingale part of $\|W_t\|^2$ is given by

$$\begin{aligned}
&\left(\left(2\left(X_t - Y_t\right)'\left(I - J_t\right)\right)\left(2\left(X_t - Y_t\right)'\left(I - J_t\right)\right)' + \left(2\left(X_t - Y_t\right)' K_t\right)\left(2\left(X_t - Y_t\right)' K_t\right)'\right) dt \\
= \quad & 4\left(X_t - Y_t\right)'\left(I - J_t - J_t' + J_t J_t' + K_t K_t'\right)\left(X_t - Y_t\right) dt \\
= \quad & 4\left(X_t - Y_t\right)'\left(2I - J_t - J_t'\right)\left(X_t - Y_t\right) dt \\
= \quad & 8\left(X_t - Y_t\right)'\left(I - J_t\right)\left(X_t - Y_t\right) dt,
\end{aligned}$$

and the differential of the bounded variation part of $\|W_t\|^2$ is given by

$$2\left(d - \operatorname{tr}\left(J_t\right)\right) dt.$$

If $\|W_t\|$ is a (deterministic) non-increasing function of $t$, we must have

$$\operatorname{tr}\left(J_t\right) \geq d \qquad \text{and} \qquad \left(X_t - Y_t\right)'\left(I - J_t\right)\left(X_t - Y_t\right) = 0$$

for all $t \geq 0$.

Denoting by $a_{ij} = a_{ij}(t)$ the entries of $J_t$, observe that

$$\operatorname{tr}\left(J_t J_t'\right) = \sum_{i,j=1}^d a_{ij}^2 \geq \sum_{i=1}^d a_{ii}^2 \geq \frac{\left(\sum_{i=1}^d a_{ii}\right)^2}{d} = \frac{\operatorname{tr}^2\left(J_t\right)}{d} \geq d,$$

with equality if and only if $J_t = I$.

On the other hand, from (3.1) it follows that $0 \leq x' J_t' J_t x \leq x'x$ for all $x \in \mathbb{R}^d$, so the eigenvalues $\lambda_i = \lambda_i(t)$ of $J_t J_t'$ satisfy $0 \leq \lambda_i \leq 1$, and therefore $\operatorname{tr}\left(J_t' J_t\right) = \sum_{i=1}^d \lambda_i \leq d$. Combining with the above we conclude that $\operatorname{tr}(J_t'J_t) = d$, and therefore $J_t = I$ for all $t \geq 0$. Equivalently, this shows that $dY_t = dX_t$ for all $t \geq 0$, or $Y_t = Y_0 - X_0 + X_t$, and we arrive at the following.

**Theorem 4.** *In the Euclidean space $\mathbb{R}^d$, $d \geq 1$, the only co-adapted coupling of Brownian motions with deterministic non-increasing distance is the translation coupling.*

As we will see later on in Theorem 10 there are distance increasing couplings on $\mathbb{R}^d$.

## 4. The 2-dimensional Sphere case, the extrinsic approach

In this section we study the couplings of Brownian motions on the unit sphere $S^2$. The primary interest is the construction of couplings for which the distance between the processes is deterministic. Using Stroock's representation of the spherical Brownian motion, we construct three different couplings, as mentioned in the introduction. In the first one the distance is decaying at an exponential rate, in the second one the distance is increasing to the diameter of the sphere $S^2$ at an exponential rate, and in the third one, which is the most interesting and intriguing, the distance is constant in time.

We collect the results on the first two couplings mentioned above in the following result, and then treat separately the latter one.

**Theorem 5.** *Fix two points $x, y \in S^2$ with $y \neq \pm x$, and consider the spherical Brownian motion $X_t$ on $S^2$ given by (2.1) with $X_0 = x$.*

a) *Let $Y_t$ be the solution to*

$$Y_t = y + \int_0^t R_s dX_s \tag{4.1}$$

*where $R_s = R_{X_s,Y_s}$ is the rotation matrix with axis $u_s = X_s \times Y_s$ and angle $\theta_s$ equal to the angle between $X_s$ and $Y_s$ (and $R_s = \pm I$ if $Y_s = \pm X_s$). Then $Y_t$ is a spherical Brownian motion on $S^2$, and*

$$\|X_t - Y_t\| = \|y - x\| e^{-t/2}, \qquad t \geq 0. \tag{4.2}$$

*In particular, $d(X_t, Y_t) = 2 \arcsin\left(\frac{1}{2}\|y - x\| e^{-t/2}\right)$ decreases exponentially fast to $0$ as $t \to \infty$.*

b) *Let $\widetilde{Y}_t$ be the solution to*

$$\widetilde{Y}_t = y - \int_0^t R_s dX_s \tag{4.3}$$

*where $R_s = R_{X_s,-\widetilde{Y}_s}$ is the rotation matrix with axis $u_s = -X_s \times \widetilde{Y}_s$ and angle $\theta_s$ equal to the angle between $X_s$ and $-\widetilde{Y}_s$ (and $R_s = \mp I$ if $\widetilde{Y}_s = \pm X_s$). Then $\widetilde{Y}_t$ is a spherical Brownian motion on $S^2$, and*

$$\left\|X_t - \widetilde{Y}_t\right\| = \sqrt{4 - \|y + x\|^2 e^{-t}}, \qquad t \geq 0. \tag{4.4}$$

*In particular, $d\left(X_t, \widetilde{Y}_t\right) = \pi - 2 \arcsin\left(\frac{1}{2}\|y + x\| e^{-t/2}\right)$ increases exponentially fast to $\pi$ as $t \to \infty$.*

*Notice that both $(X_t, Y_t)$ and $\left(X_t, \widetilde{Y}_t\right)$ are both Markovian couplings. In fact they are diffusions on $S^2 \times S^2$.*

*Proof.* Using (2.1), we first write

$$dY_t = R_t dX_t = R_t \left(I - X_t X_t'\right) dB_t - R_t X_t dt.$$

By definition, $R_t$ is a unitary matrix and $R_t X_t = Y_t$ all $t \geq 0$, from which we obtain

$$\begin{aligned} dY_t &= \left(R_t - R_t X_t X_t'\right) dB_t - R_t X_t dt = \left(I - (R_t X_t)(R_t X_t)'\right) R_t dB_t - R_t X_t dt = \left(I - Y_t Y_t'\right) R_t dB_t - Y_t dt \\ &= \left(I - Y_t Y_t'\right) d\widetilde{B}_t - Y_t dt, \end{aligned}$$

where $\widetilde{B}_t = \int_0^t R_s dB_s$ is readily seen to be a 3-dimensional Brownian motion. Using again Stroock's characterization of spherical Brownian motion, the first claim follows.

To prove the second claim, we apply the Itô formula to the function $f(z) = z'z$ and to the process $Z_t = Y_t - X_t$. We get

$$d\|Z_t\|^2 = 2Z_t' dZ_t + \sum_{i=1}^{3} d\langle Z^i\rangle_t.$$

Next, we'll show that $\|Z_t\|^2$ is a process of bounded variation. To do this, we write

$$\begin{aligned} dZ_t &= d(Y_t - X_t) = (R_t - I)\,dX_t = (R_t - I)\left(I - X_t X_t'\right) dB_t - (R_t - I)\,X_t dt \\ &= \left(R_t - R_t X_t X_t' - I + X_t X_t'\right) dB_t - (Y_t - X_t)\,dt = M_t dB_t - Z_t dt, \end{aligned}$$

where $M_t = R_t - R_t X_t X_t' - I + X_t X_t'$. Combining with the above, we get

$$d\|Z_t\|^2 = 2Z_t' M_t dB_t - 2Z_t' Z_t dt + \sum_{i=1}^{3} d\langle Z^i\rangle_t, \tag{4.5}$$

and in order to prove the claim it suffices to show that $Z_t' M_t \equiv 0$. Notice that

$$\begin{aligned} Z_t' M_t &= (R_t X_t - X_t)'\left(R_t - R_t X_t X_t' - I + X_t X_t'\right) \\ &= X_t' R_t' R_t - X_t' R_t' R_t X_t X_t' - X_t' R_t' + X_t' R_t' X_t X_t' - X_t' R_t + X_t' R_t X_t X_t' + X_t' - X_t' X_t X_t' \\ &= X_t' - X_t' - X_t' R_t' + X_t' R_t' X_t X_t' - X_t' R_t + X_t' R_t X_t X_t' + X_t' - X_t' = X_t'\left(R_t + R_t'\right)\left(X_t X_t' - I\right). \end{aligned}$$

Using the representation in (2.2) for $R_t$, since $\left([u_t]_\times\right)' = (Y_t X_t' - X_t Y_t')' = -[u_t]_\times$ and $(u_t \otimes u_t)' = (u_t u_t')' = u_t \otimes u_t$, we obtain:

$$\begin{aligned} Z_t' M_t &= 2X_t'\left(\cos\theta_t I + \frac{1}{1+\cos\theta_t}(X_t \times Y_t)(X_t \times Y_t)'\right)\left(X_t X_t' - I\right) \\ &= 2\cos\theta_t\left(X_t' X_t X_t' - X_t'\right) + \frac{2}{1+\cos\theta_t} X_t'(X_t \times Y_t)(X_t \times Y_t)'\left(X_t X_t' - I\right) = 0, \end{aligned}$$

where in the last equality we used $X_t' X_t = \|X_t\|^2 = 1$ and $X_t'(X_t \times Y_t) \equiv 0$ (the vector $X_t \times Y_t$ being orthogonal to $X_t$). It thus follows that $Z_t' M_t \equiv 0$ as we claimed, and therefore $\|Z_t\|^2$ is a process of bounded variation, given by

$$d\|Z_t\|^2 = -2\|Z_t\|^2 dt + \sum_{i=1}^{3} d\langle Z^i\rangle_t. \tag{4.6}$$

Finally, note that by using (4.5) we can write the last term in the above equation as

$$\sum_{i=1}^{3} d\langle Z^i\rangle_t = \operatorname{tr}\left(M_t M_t'\right) dt,$$

and since $X_t$ is on the unit sphere (so $X_t' X_t = 1$ and $(I - X_t X_t')^2 = I - X_t X_t'$), we can continue with

$$\begin{aligned} \operatorname{tr}(M_t M_t') &= \operatorname{tr}((R_t - I)(I - X_t X_t')^2(R_t' - I)) = \operatorname{tr}((R_t' - I)(R_t - I)(I - X_t X_t')) \\ &= \operatorname{tr}((2I - R_t' - R_t)(I - X_t X_t')) = 2\operatorname{tr}(I - X_t X_t') - \operatorname{tr}((R_t' + R_t)(I - X_t X_t')) \\ &= 6 - 2X_t' X_t - 2\operatorname{tr}(R_t(I - X_t X_t')) = 4 - 2\operatorname{tr}(R_t) + 2\operatorname{tr}(R_t X_t X_t') = 4 - 2\operatorname{tr}(R_t) + 2Y_t' X_t, \end{aligned} \tag{4.7}$$

where in passing to the last line we used that $R_t X_t = Y_t$. Using the fact that the trace of the rotation matrix $R_t$ equals the sum $1 + 2\cos\theta_t$ of its eigenvalues (recall that by construction the angle $\theta_t$ of rotation of $R_t$ is the angle between $X_t$ and $Y_t$), we can conclude that

$$\operatorname{tr}(M_t M_t') = 4 - 2(1 + 2Y_t' X_t) + 2Y_t' X_t = 2 - 2Y_t' X_t = \|X_t - Y_t\|^2 = \|Z_t\|^2.$$

Wrapping things up, we obtained

$$d\|Z_t\|^2 = -\|Z_t\|^2 dt, \qquad t \geq 0.$$

Setting $\tau = \inf\{t \geq 0 : Z_t = 0\}$, the above can be solved as an ordinary differential equation for $t < \tau = \tau(\omega)$ for any path $\omega \in \Omega$, and we obtain the solution

$$\|Z_t\| = \|Y_t - X_t\| = \|y - x\| e^{-t/2}, \qquad t < \tau. \tag{4.8}$$

In particular we see that for any $x \neq y$ we have $Z_t \neq 0$ a.s. for all $t \geq 0$, and therefore $\tau = \infty$ a.s. This shows that

$$\|Y_t - X_t\| = \|y - x\| e^{-t/2}, \qquad t \geq 0,$$

which concludes the proof of the first part of the theorem.

To prove the second part of the theorem, note that if $\widetilde{Y}_t$ solves (4.3), then $Y_t := -\widetilde{Y}_t$ solves (4.1) with $y$ replaced by $-y$ (the process $Y_t$ starts at $-y$ instead of $y$). If $C_t$ denotes the circle on $S^2$ of radius 1 and passing through $X_t$ and $Y_t$ (since $x \neq -y$, by the previous proof we have that $X_t \neq Y_t$ for all $t \geq 0$, and thus $C_t$ is well defined), it follows that $\widetilde{Y}_t = -Y_t \in C_t$ for all $t \geq 0$. The second part of the theorem follows now easily from the first part using simple geometric considerations. □

We now proceed to showing the existence of a fixed-distance coupling of Brownian motions on $S^2$, that is a Markovian coupling $(X_t, Y_t)$ of spherical Brownian motions for which the distance $d(X_t, Y_t)$ is constant for all times $t \geq 0$.

Assume such a coupling exists, and that $X_t$ and $Y_t$ are given by

$$dX_t = \left(I - X_t X_t'\right) dB_t - X_t dt \qquad \text{and} \qquad dY_t = \left(I - Y_t Y_t'\right) dW_t - Y_t dt, \tag{4.9}$$

where $B_t$ and $W_t$ are the driving 3-dimensional Brownian motions, and $X_0 = x, Y_0 = y \in S^2$.

By a result on co-adapted couplings of free Brownian motions (assuming that the coupling is co-adapted, see Lemma 6 in [16]), there exist $3 \times 3$ matrices $J_t$ and $K_t$ with

$$J_t J_t' + K_t K_t' = I \tag{4.10}$$

and a 3-dimensional Brownian motion $C_t$ independent of $B_t$ such that

$$dW_t = J_t dB_t + K_t dC_t. \tag{4.11}$$

The idea is now very simple. We want to find the matrix-valued processes $J_t$ and $K_t$ such that the distance between $X_t$ and $Y_t$ does not change with time. The theorem below shows that such a construction is possible, and that in fact the resulting coupling is not only co-adapted, but also a Markovian coupling.

**Theorem 6.** *For any points $x, y \in S^2$, there exists a fixed-distance Markovian coupling of Brownian motions on the 2-dimensional unit sphere $S^2$ starting at $x$ and $y$. As it turns out, the process $(X_t, Y_t)$ is actually a diffusion on $S^2 \times S^2$.*

*Proof.* The claim is trivial if $x = \pm y$, so we may assume $x \neq \pm y$.

Denoting $Z_t = X_t - Y_t$, $U_t = I - X_t X_t'$ and $V_t = I - Y_t Y_t'$ (note that $U_t$ and $V_t$ are symmetric matrices, with $U_t^2 = U_t$ and $V_t^2 = V_t$), and using the above equations we obtain

$$dZ_t \quad = \quad U_t dB_t - V_t dW_t - Z_t dt = (U_t - V_t J_t)\, dB_t - V_t K_t dC_t - Z_t dt.$$

Ito's formula gives after expansion and rearrangements that

$$d\|Z_t\|^2 = 2Z_t' dZ_t + \sum_{i=1}^{3} d\langle Z^i \rangle_t = 2M_t dB_t + 2N_t dC_t - 2\|X_t - Y_t\|^2 dt + \sum_{i=1}^{3} d\langle Z^i \rangle_t$$

with $M_t = -X_t' V_t J_t - Y_t' U_t$ and $N_t = -X_t' V_t K_t$.

The fact that $B_t$ and $C_t$ are independent Brownian motions allows us to compute the quadratic variation of $\|Z_t\|^2$ as follows:

$$\begin{aligned} \frac{1}{4} d\left\langle \|Z\|^2 \right\rangle_t &= \left(M_t M_t' + N_t N_t'\right) dt = \left(X_t' V_t \left(J_t J_t' + K_t K_t'\right) V_t' X_t + X_t' V_t J_t U_t' Y_t + Y_t' U_t J_t' V_t' X_t + Y_t' U_t Y_t\right) dt \\ &= \left(X_t' V_t X_t + X_t' V_t J_t U_t Y_t + Y_t' U_t J_t' V_t X_t + Y_t' U_t Y_t\right) dt. \end{aligned}$$

Note that $X_t'V_tX_t = X_t'(I - Y_tY_t')X_t = X_t'X_t - X_t'Y_tY_t'X_t = 1 - c_t^2$, where $c_t = Y_t'X_t$, and similarly $Y_t'V_tY_t = 1 - c_t^2$. Since $X_t'V_tJ_tU_tY_t$ is a real number, it equals its transpose which is $Y_t'U_tJ_t'V_tX_t$. Keeping in mind that $X_t'Y_t = Y_t'X_t = c_t$, we also get

$$\begin{aligned} X_t'V_tJ_tU_tY_t &= X_t'(I - Y_tY_t')J_t(I - X_tX_t')Y_t = (X_t' - c_tY_t')J_t(Y_t - c_tX_t) \\ &= X_t'J_tY_t - c_tX_t'J_tX_t - c_tY_t'J_tY_t + c_t^2Y_t'J_tX_t, \end{aligned}$$

and therefore

$$\frac{1}{4}d\left\langle \|Z\|^2 \right\rangle_t = 2(1 - c_t^2 + X_t'J_tY_t - c_tX_t'J_tX_t - c_tY_t'J_tY_t + c_t^2Y_t'J_tX_t)dt.$$

If $\|Z_t\|^2$ is to be a constant process, then its quadratic variation must be identically zero, or

$$1 - c_t^2 + X_t'J_tY_t - c_tX_t'J_tX_t - c_tY_t'J_tY_t + c_t^2Y_t'J_tX_t = 0, \tag{4.12}$$

and its bounded variation part must also be identically zero. To see what the latter equation is, from (4.12) we gain

$$\begin{aligned} -2\|X_t - Y_t\|^2 dt + \sum_{i=1}^3 d\langle Z^i\rangle_t &= \left(-2\|X_t - Y_t\|^2 + \mathrm{tr}\left((U_t - V_tJ_t)(U_t - V_tJ_t)' + (V_tK_t)(V_tK_t)'\right)\right)dt \\ &= \left(-4(1 - c_t) + \mathrm{tr}\left(U_t - U_tJ_t'V_t' - V_tJ_tU_t' + V_t\left(J_tJ_t' + K_tK_t'\right)V_t'\right)\right)dt \\ &= \left(4c_t - 2\mathrm{tr}\left(V_tJ_tU_t\right)\right)dt, \end{aligned}$$

which continues with

$$\begin{aligned} \mathrm{tr}\left(V_tJ_tU_t\right) &= \mathrm{tr}\left(J_tU_tV_t\right) = \mathrm{tr}\left(J_t\left(I - X_tX_t'\right)\left(I - Y_tY_t'\right)\right) = \mathrm{tr}\left(J_t - J_tX_tX_t' - J_tY_tY_t' + c_tJ_tX_tY_t'\right) \\ &= \mathrm{tr}\left(J_t\right) - \mathrm{tr}\left(J_tX_tX_t'\right) - \mathrm{tr}\left(J_tY_tY_t'\right) + c_t\mathrm{tr}\left(J_tX_tY_t'\right) = \mathrm{tr}\left(J_t\right) - X_t'J_tX_t - Y_t'J_tY_t + c_tY_t'J_tX_t, \end{aligned}$$

finally arriving at

$$X_t'J_tX_t + Y_t'J_tY_t - c_tY_t'J_tX_t = \mathrm{tr}\left(J_t\right) - 2c_t. \tag{4.13}$$

The above shows that we can construct a co-adapted fixed-distance coupling of Brownian motions on $S^2$ iff we can find matrices satisfying (4.10), (4.12) and (4.13).

We can go one step further and simplify (4.12). Indeed, because of (4.13), it is easy to see that equation (4.12) is equivalent to

$$1 - c_t^2 + X_t'J_tY_t - c_t\left(\mathrm{tr}\left(J_t\right) - 2c_t\right) = 0.$$

Consequently, the existence of a fixed-distance co-adapted coupling of Brownian motions on $S^2$ is equivalent to solving for $J_t$ and $K_t$ from the following system

$$\left\{ \begin{array}{ll} X_t'J_tY_t = -c_t^2 + c_t\mathrm{tr}\left(J_t\right) - 1 & (\text{or } X_t'V_tJ_tU_tY_t = c_t^2 - 1) \\ X_t'J_tX_t + Y_t'J_tY_t - c_tY_t'J_tX_t = \mathrm{tr}\left(J_t\right) - 2c_t & (\text{or } \mathrm{tr}\left(V_tJ_tU_t\right) = 2c_t) \\ J_tJ_t' + K_tK_t' = I. & \end{array} \right.$$

To find a solution of this system, we are easing a little bit the notations by dropping for the moment the dependence on $t$. Hence, given two vectors $X, Y$ on the unit sphere we want to find two $3 \times 3$ matrices $J$ and $K$ such that

$$\begin{cases} X'JY = c\,\mathrm{tr}(J) - 1 - c^2 \\ X'JX + Y'JY - cY'JX = \mathrm{tr}(J) - 2c \\ JJ' + KK' = I \end{cases} \tag{4.14}$$

where $c = X'Y$. The first two equations above involve only $J$. Assuming that we can determine $J$ which satisfies these equations, from the third equation of the system we can find the matrix $K$ such that $KK' = I - JJ'$ if and only if $JJ' \leq I$ in the operator sense, i.e. $\xi'JJ'\xi \leq 1$ for any unit vector $\xi \in \mathbb{R}^3$, or equivalently $\|J'\xi\| \leq 1$. The latter condition is the same as the operator norm of $J'$ is less than $1$, or $\|J\| \leq 1$, since the operator norm of $J$ and $J'$ are the same.

Assume now that $X, Y \in S^2$ with $X \neq \pm Y$ are fixed. We can find an orthogonal matrix $O_{X,Y}$ such that

$$O_{X,Y} e_1 = X \text{ and } O_{X,Y}(ce_1 + \sqrt{1-c^2} e_2) = Y$$

where here $(e_i)_{i=1,2,3}$ is the standard basis in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and } e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

One way of choosing such a matrix $O_{X,Y}$ is for example by taking

$$O_{X,Y}[e_1, ce_1 + \sqrt{1-c^2} e_2, \sqrt{1-c^2} e_3] = [X, Y, X \times Y],$$

where $[X, Y, Z]$ denotes the matrix whose columns are the vectors $X, Y, Z$. It is worth mentioning that if the matrix $O_{X,Y}$ is to be orthogonal, then it has to map $e_3$ into an unitary vector which is collinear to $X \times Y$, which in this case gives $O_{X,Y} e_3 = \pm \frac{1}{\sqrt{1-c^2}} X \times Y$, so there are essentially two choices for the matrix $O_{X,Y}$.

Computing the inverse

$$[e_1, ce_1 + \sqrt{1-c^2} e_2, \sqrt{1-c^2} e_3]^{-1} = \begin{bmatrix} 1 & -\frac{c}{\sqrt{1-c^2}} & 0 \\ 0 & \frac{1}{\sqrt{1-c^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-c^2}} \end{bmatrix},$$

we obtain an explicit formula for $O_{X,Y}$

$$O_{X,Y} = \begin{bmatrix} x_1 & y_1 & x_2 y_3 - y_2 x_3 \\ x_2 & y_2 & x_3 y_1 - y_3 x_1 \\ x_3 & y_3 & x_1 y_2 - y_1 x_2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{c}{\sqrt{1-c^2}} & 0 \\ 0 & \frac{1}{\sqrt{1-c^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-c^2}} \end{bmatrix} = \begin{bmatrix} x_1 & \frac{-cx_1+y_1}{\sqrt{1-c^2}} & \frac{x_2y_3-y_2x_3}{\sqrt{1-c^2}} \\ x_2 & \frac{-cx_2+y_2}{\sqrt{1-c^2}} & \frac{x_3y_1-y_3x_1}{\sqrt{1-c^2}} \\ x_3 & \frac{-cx_3+y_3}{\sqrt{1-c^2}} & \frac{x_1y_2-y_1x_2}{\sqrt{1-c^2}} \end{bmatrix}. \tag{4.15}$$

Note that since $X \neq \pm Y$, $c \neq \pm 1$, so the matrix $O_{X,Y}$ is well defined.

Finding a solution $J$ to the system (4.14) is equivalent to finding a solution

$$\widetilde{J} = O'_{X,Y} J O_{X,Y} \text{ and } \widetilde{K} = O'_{X,Y} K O_{X,Y}$$

to the system obtained from (4.14) by replacing $X$ by $e_1$, and $Y$ by $ce_1 + \sqrt{1-c^2} e_2$, which becomes

$$\begin{cases} ce_1' \widetilde{J} e_1 + \sqrt{1-c^2} e_1' \widetilde{J} e_2 = c \operatorname{tr}(\widetilde{J}) - 1 - c^2 \\ e_1' \widetilde{J} e_1 + c\sqrt{1-c^2} e_1' \widetilde{J} e_2 + (1-c^2) e_2' \widetilde{J} e_2 = \operatorname{tr}(\widetilde{J}) - 2c \\ \widetilde{J}\widetilde{J}' + \widetilde{K}\widetilde{K}' = I. \end{cases}$$

Now let

$$\widetilde{J} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix},$$

which turns the first two equations of the above system into

$$\begin{cases} \sqrt{1-c^2}\alpha_2 - c\beta_2 - c\gamma_3 = -1 - c^2 \\ c\sqrt{1-c^2}\alpha_2 - c^2\beta_2 - \gamma_3 = -2c. \end{cases} \tag{4.16}$$

This is a system of two equations with three unknown which can be reduced to

$$\begin{cases} \beta_2 = \frac{\sqrt{1-c^2}\alpha_2 + 1}{c} \\ \gamma_3 = c. \end{cases}$$

Of course the case $c = 0$ needs to be treated separately, in which case, it is obvious that $\alpha_2 = -1$ and $\gamma_3 = 0$.

In the case $c \neq 0$, the simplest matrix $\widetilde{J}$ which satisfies the above conditions is the one whose entries are all 0 except for $\alpha_2$, $\beta_2$, and $\gamma_3$, so we may try

$$\widetilde{J} = \begin{bmatrix} 0 & \alpha_2 & 0 \\ 0 & \frac{\sqrt{1-c^2}\alpha_2+1}{c} & 0 \\ 0 & 0 & c \end{bmatrix}.$$

The main restriction now is that we want the operator norm of $\widetilde{J}$ to be at most 1. Because of the block diagonal structure, this is equivalent to

$$\alpha_2^2 + \frac{(\sqrt{1-c^2}\alpha_2+1)^2}{c^2} \leq 1$$

or

$$(\alpha_2 + \sqrt{1-c^2})^2 \leq 0,$$

whose solution is $\alpha_2 = -\sqrt{1-c^2}$, and consequently

$$\widetilde{J} = \begin{bmatrix} 0 & -\sqrt{1-c^2} & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

This matrix now is well defined also for $c = 0$ and is consistent with the solutions provided by the system (4.16).

For the above choice of $\widetilde{J}$ we need to find $\widetilde{K}$ such that

$$\widetilde{J}\widetilde{J}' + \widetilde{K}\widetilde{K}' = I,$$

which reduces to

$$\widetilde{K}\widetilde{K}' = \begin{bmatrix} c^2 & c\sqrt{1-c^2} & 0 \\ c\sqrt{1-c^2} & 1-c^2 & 0 \\ 0 & 0 & 1-c^2 \end{bmatrix}.$$

There are several possible choices here, one of them being

$$\widetilde{K} = \begin{bmatrix} 0 & c & 0 \\ 0 & \sqrt{1-c^2} & 0 \\ 0 & 0 & \sqrt{1-c^2} \end{bmatrix}.$$

Going back to initial problem, we obtain the solution

$$J = O_{X,Y}\widetilde{J}O'_{X,Y} \text{ and } K = O_{X,Y}\widetilde{K}O'_{X,Y}.$$

The only possible problem with this choice of the matrices $J$ and $K$ is that when the particles $X$ and $Y$ get close or antipodal ($X = \pm Y$), the above matrices are undefined because $O_{X,Y}$ does not. However, this does not happen, since by hypothesis $x \neq \pm y$, and with the above choices of $J$ and $K$ the Brownian motions $X_t$ and $Y_t$ are at a fixed-distance (the initial distance).

Finally, since $(X_t, Y_t)$ solves a stochastic differential equation and the matrices $J_t$ and $K_t$ are actually functions of $(X_t, Y_t)$, this means that the process $(X_t, Y_t)$ is in fact a diffusion on $S^2 \times S^2$. This is in fact stronger than mere Markovianity. □

**Remark 7.** *It is tempting to extend this argument to higher dimensional spheres. If we follow the same argument we do not have to change anything up to* (4.14). *The attempt on solving* (4.14) *was based on arranging the vectors $X, Y$ in a certain position, in other words, make $X$ for instance to be $e_1$ and $Y$ a linear combination of $e_1$ and $e_2$. Since there is essentially (up to a sign choice) a unique perpendicular unit vector to both $X$ and $Y$, the condition that $O_{X,Y}$ sends this into $e_3$ determines the matrix $O_{X,Y}$ perfectly well. In higher dimensions this becomes an issue because there is no canonical choice of the matrix $O_{X,Y}$. Indeed, given two vectors $X, Y$ it is not clear that one can produce a number of vectors which depend smoothly on $X, Y$ and be a basis of the orthogonal complement of the span of $X, Y$. In more abstract terms, if $V_{k,n}$ is the Stiefel manifold of $k$ orthogonal frames ($k \geq 2$) in $\mathbb{R}^n$, then our problem becomes equivalent to the problem of finding a cross section of the projection $V_{k,n} \to V_{2,n}$. The projection used here is sending the frame $f_1, f_2, \ldots, f_k$ into $f_1, f_2$. It is know that this is possible (cf.* [14, Theorem 1.7]*)* *if and only if* $n = 3$ *and* $k = 3$*, and this shows that the proof above works essentially only for the* 2*-dimensional sphere.*

*For the higher dimensional spheres, we are going to use a different approach. So far, we have only used the extrinsic approach which is very versatile in the present context, but could become a weakness when one wants to extend it to other manifolds.*

**Remark 8.** *Without much extra work one can refine the result in Theorem 6 and show that for any* $0 \leq k \leq 1$ *and* $x, y \in S^2$*, there is a Markovian coupling* $(X_t, Y_t)$ *starting at* $(x, y)$ *such that* $\|X_t - Y_t\| = e^{-kt/2}\|x - y\|$ *for all* $t \geq 0$*.*

*If* $k < 0$*, then there is a coupling* $(X_t, Y_t)$ *initiated at* $(x, y)$ *such that* $\|X_t - Y_t\| = e^{-kt/2}\|x - y\|$ *but only for* $0 \leq t \leq \delta$ *where* $\delta$ *is a constant determined by* $k$ *and* $\|x - y\|$*. Notice that the distance increases exponentially fast in the case* $k < 0$*, and because of the compactness of* $S^2$ *this coupling exists only for short time.*

*The proof is just a straightforward refinement of the one of Theorem 6 and is left to the reader. An interesting feature of the proof is that the upper limit of* $k$ *for which we can get the exponential distance is* $k = 1$*. This is perhaps a reflection of the fact that the curvature of* $S^2$ *is actually* 1*.*

**Remark 9.** *After we wrote this paper, the second author talked to Thierry Léve who gave us a different and nice construction of the fixed-distance coupling which we briefly describe now. Consider* $O(3)$*, the set of* $3\times 3$ *orthogonal matrices, and denote for each point* $x \in S^2$ *the map* $\pi_x : O(3) \to S^2$ *given by* $\pi_x(A) = Ax$*. On* $O(3)$ *we take the standard left-right invariant metric and the Riemannian structure associated to this. Now, if* $\Delta^{O(3)}$ *and* $\Delta^{S^2}$ *denote the Laplacians on* $O(3)$ *and* $S^2$ *respectively, it turns out that* $\Delta^{O(3)}(f \circ \pi_x) = \Delta^{S^2}(f)$ *whose consequence is that if* $Z_t$ *is a Brownian motion on* $O(3)$*, then* $Z_t x$ *is a Brownian motion on* $S^2$*. In this way, if we pick two points* $x, y$ *on* $S^2$*, then* $X_t = Z_t x$ *and* $Y_t = Z_t y$ *is a fixed distance coupling on* $S^2$*.*

## 5. An Intrinsic Approach on Manifolds of Constant Curvature

From the geometric point of view, the constructions in the previous sections are extrinsic, which means that the manifold in discussion ($S^2$) is imbedded into another manifold ($\mathbb{R}^3$). The intrinsic point of view is to study the manifold with reference to its own Riemannian structure and not by embedding it into another one. Since the Brownian motion on a manifold is essentially an intrinsic object, it is natural to try to find couplings which are defined in terms of the intrinsic structure. This is what we want to achieve in this section.

We start with a $d$-dimensional Riemannian manifold $M$ and we will use the notations introduced in Section 4. On $M$, one can construct the Brownian motion as the solution to a martingale problem associated to the Laplacian. For more details, we refer the reader to [12] and [25].

What we intend to construct is a coupling following the line of ideas from [12, Section 6.5] where the mirror coupling is discussed. Keeping the notation from [12], we want to define the coupling as a solution to a certain stochastic differential system at the level of orthonormal frame bundle. If $U_0$ is a given orthonormal frame bundle at $x_0$ and $V_0 = O_{x_0,y_0}U_0$ is an orthonormal frame bundle at $y_0$, the system we consider is

$$\begin{cases} dU_t = \sum_{i=1}^d H_i(U_t) \circ dW_t^i \\ dV_t = \sum_{i=1}^d H_i(V_t) \circ dB_t^i \\ dB_t = V_t^{-1} O_{X_t Y_t} U_t dW_t \\ X_t = \pi U_t \\ Y_t = \pi V_t \end{cases} \tag{5.1}$$

where $O_{x,y}$ is an isometry from $T_x M$ into $T_y M$ to be chosen later. Notice that in order to ensure that $B_t$ is also a Brownian motion on $\mathbb{R}^n$, we need to make sure that the map $O_{x,y}$ is actually an isometry. This in turn guarantees that $X_t$ and $Y_t$ are Brownian motions on the base manifold $M$ starting at $x_0$ and $y_0$. In what follows we use the notation

$$\mathfrak{e}_i(t) = \pi_* H_i(U_t) \tag{5.2}$$

which is the projection of the vector $H_i(U_t)$ onto the base tangent space $T_{X_t}M$ whose common interpretation is as the parallel transport of $U_0 e_i$ along the Brownian path $X$ from $T_{x_0}M$ to $T_{X_t}M$.

The generator of the diffusion $(X_t, Y_t)$ up to the first time $t$ for which $O_{X_t,Y_t}$ becomes undefined is given by

$$\mathcal{L}_{x,y} = \Delta_x + \Delta_y + 2\langle O_{x,y}E_i, F_j\rangle E_i F_j,$$

where $E_i$ is an orthonormal basis at $x$ and $F_i$ is an orthonormal basis at $y$. As one can check this is consistent because the definition of $\mathcal{L}_{x,y}$ does not depend on the choices of the orthonormal basis $E_i$ and $F_j$. Perhaps a little better suited to Itô's formula is the writing

$$\mathcal{L}_{x,y} = \sum_{i=1}^{n}(E_i + O_{x,y}E_i)^2$$

with the understanding that the action of $E_i$ on a function $f(x,y)$ is with respect to the variable $x$, while $O_{x,y}E_i$ acts on $f$ as the derivative with respect to $y$.

The first natural thing to look at in the context of couplings is the distance function between the processes. Thinking of $d(x,y)$ as a function of two variables and using Ito's formula for the semimartingale decomposition of $\rho_t = d(X_t, Y_t)$, one obtains

$$d\rho_t = ((\mathfrak{e}_i(t) + O_{X_t,Y_t}\mathfrak{e}_i(t))\rho)(X_t, Y_t)dB_t^i + \frac{1}{2}(\mathcal{L}\rho)(X_t, Y_t)dt \tag{5.3}$$

which is well defined up to the first time $t$ when either $X_t, Y_t$ get at each other's cut-locus or $O_{X_t,Y_t}$ ceases to be defined. As it is known, e.g. [12, Theorem 6.6.2], for the mirror coupling we have a better result, namely,

$$d\rho_t = ((\mathfrak{e}_i(t) + O_{X_t,Y_t}\mathfrak{e}_i(t))\rho)(X_t, Y_t)dB_t^i + \frac{1}{2}(\mathcal{L}\rho)(X_t, Y_t)dt - dL_t$$

where the process $L_t$ is a positive non-decreasing process which increases only at the cut-locus.

In our framework this should follow in a similar manner as the argument of the proof of [12, Theorem 6.6.2] for the mirror coupling. However, instead of appealing to an extension of this argument, we want to use a soft version of that argument for the construction of the coupling here. Assume that we have a smooth choice of $O_{x,y}$ for all $x, y$ which are not at each other cut-locus.

The key step in the definition of the process $L_t$ above is the following. Take a small $\epsilon > 0$ and let $C_\epsilon$ be the $\epsilon$-neighborhood of the cut-locus $Cut$. Then, if we start the coupling from two points which are within the injectivity radius and run it until it hits $C_\epsilon$, and then we continue the process with two independent copies until the joint process exits $C_{2\epsilon}$. At this point we resume the coupling given above until it hits again $C_\epsilon$, at which point we switch again to two independent Brownian motions until the joint process leaves $C_{2\epsilon}$, and so on. The main point in the proof of the existence of $L_t$ is to show that one can let $\epsilon$ go to zero and that the time spent in $C_{2\epsilon}$ goes to 0. We do not need this refined result here, we just need a process which continues to have the marginals as Brownian motions, and outside of a small neighborhood of the cut-locus runs according to (5.1). This extension is discussed by Cranston in [9].

We will refer to this construction here as the *extension beyond the cut-locus* and we will use it from now on as soon as the rotation matrix $O_{x,y}$ is defined up to a complement of $C_\epsilon$ for some $\epsilon > 0$. Notice that with this construction we do not need to define $O_{x,y}$ for $(x,y) \in C_\epsilon$. Also, it can be shown that this construction is well defined for all times $t \geq 0$ if $O_{x,y}$ depends smoothly on $x, y$ outside $C_\epsilon$. For instance, the Brownian motion spends a certain amount of time inside $C_\epsilon$ due to the curvature condition, and also the coupling spends a certain amount of time outside $C_\epsilon$, and such it is not hard to check that the coupling does not explode in finite time. This is relatively standard and some of these arguments are outlined in Subsection 6.4.

Going back to the equation (5.3), we notice that the first variation formula provides us with a way of computing the martingale part via

$$((\mathfrak{e}_i(t) + O_{X_t,Y_t}\mathfrak{e}_i(t))\rho)(X_t, Y_t) = \langle O_{X_t,Y_t}\mathfrak{e}_i(t), \dot{\gamma}_t(\rho_t)\rangle - \langle \mathfrak{e}_i(t), \dot{\gamma}_t(0)\rangle, \tag{5.4}$$

where $\gamma_t$ is the minimizing geodesic curve joining $X_t$ to $Y_t$ and running at unit speed. If we want the coupling to have a deterministic distance, we need the martingale part to be 0 and thus we need to enforce that

$$\langle O_{X_t,Y_t}\mathfrak{e}_i(t), \dot\gamma_t(\rho_t)\rangle - \langle \mathfrak{e}_i(t), \dot\gamma_t(0)\rangle = 0. \tag{5.5}$$

Since $\mathfrak{e}_i(t)$ is an orthonormal basis at $T_{X_t}M$, this means that the orthogonal transformation $O_{X_t,Y_t}$ preserves the tangential and vertical spaces in the geodesic direction along $\gamma_t$.

The identification of the bounded variation part of (5.4) is also pretty standard and follows the same proof as the one in [12]. The result is that

$$(\mathcal{L}\rho)(X_t,Y_t) = \sum_{i=1}^d \mathcal{I}(J_i,J_i), \tag{5.6}$$

where $\mathcal{I}$ is the index form defined in (2.4) and $J_i$ is the Jacobi field along the geodesic $\gamma_t$ with the boundary conditions $\mathfrak{e}_i(t)$ at $\gamma(0)$ and $O_{X_t,Y_t}\mathfrak{e}_i(t)$ at $\gamma(\rho_t)$.

In fact, the sum in (5.6) is based on the initial values of the Jacobi fields $J_i$ which are given in terms of $\mathfrak{e}_i$. It turns out that if we pick any orthonormal basis at $T_{X_t}M$, say $E_i$, and let $J_i$ be the Jacobi field $J_i$ determined by the condition that at the endpoints equals $E_i$, respectively $O_{X_t,Y_t}E_i$, then the sum in (5.6) does not change. This observation is useful for later computations.

Among the candidates to the role of the isometry $O_{x,y}$ one is the parallel transport along the minimizing geodesic from $x$ to $y$, and the resulting coupling is called *synchronous coupling*.

The other choice which fits into the picture is the one when $O_{x,y}$ preserves the tangential component of the geodesic from $x$ to $y$, but changes the sign of the vertical component after performing the parallel transport. Geometrically this is a version of *perverse coupling* and we will refer it so. With this choice, perpendicular to the geodesic the particles move in opposite directions.

More precisely, let $\tau_{x,y}$ be the parallel transport of $T_xM$ into $T_yM$ along the minimizing geodesic $\gamma$ and let $T_xM = \mathbb{R}\dot\gamma(0) + T_{xy}^\perp$ be the orthogonal decomposition of $T_xM$ into the geodesic direction and the perpendicular direction. Similarly let $T_y = \mathbb{R}\dot\gamma(\rho) + T_{yx}^\perp$ with $\rho = d(x,y)$. The two choices described above are given by

$$O_{x,y}\dot\gamma(0) = \dot\gamma(\rho) \text{ and } O_{x,y}\xi = \tau_{x,y}\xi, \tag{5.7}$$

respectively by

$$O_{x,y}\dot\gamma(0) = \dot\gamma(\rho) \text{ and } O_{x,y}\xi = -\tau_{x,y}\xi, \tag{5.8}$$

for any $\xi \in T_{xy}^\perp$.

The first result of this section summarizes the main properties of the coupling in the case of constant curvature manifolds, as follows.

**Theorem 10.** *Let $M$ be a complete $d$-dimensional Riemannian manifold of constant sectional curvature $r$. For simplicity consider only the cases $r = -1$, $0$ or $1$, the general case following by a scaling argument.*

*If the starting points $x_0, y_0$ are chosen such that $\rho_0 < i(M)/2$, then the following hold.*

a) *For the choice of $O_{x,y}$ as in (5.7), the coupling of the Brownian motions satisfies the property that*

$$\begin{cases} \text{if } r=-1, & \rho_t \ge \rho_0 \text{ for all } t\ge 0 \\ \text{if } r=0, & \rho_t = \rho_0 \text{ for all } t\ge 0 \\ \text{if } r=1, & 0<\rho_t \le Ce^{-(d-1)t/2} \text{ for all } t\ge 0 \text{ and some constant } C>0. \end{cases} \tag{5.9}$$

b) *For the choice of $O_{x,y}$ as in (5.8), in all cases,*

$$\rho_t \ge \rho_0 \text{ for all } t \ge 0. \tag{5.10}$$

*Moreover, in the case of the model spaces, namely the hyperbolic space ($r=-1$), the sphere ($r=1$), and the plane ($r=0$), for any starting points $x_0 \neq y_0$ which are not at each other's cut-locus, the following hold true.*

c) *For the choice of $O_{x,y}$ as in (5.7),*

$$\begin{cases} \text{if } r=-1, & \rho_t = 2\operatorname{arcsinh}(e^{(d-1)/2}\sinh(\rho_0/2)) \text{ for all } t\ge 0 \\ \text{if } r=0, & \rho_t = \rho_0 \text{ for all } t\ge 0 \\ \text{if } r=1, & \rho_t = 2\arcsin(e^{-(d-1)t/2}\sin(\rho_0/2)) \text{ for all } t\ge 0. \end{cases} \tag{5.11}$$

d) *For the choice of $O_{x,y}$ as in (5.8),*

$$\begin{cases} \textit{if } r=-1, & \rho_t = 2\text{arccosh}(e^{(d-1)t/2}\cosh(\rho_0/2)) \textit{ for all } t\ge 0 \\ \textit{if } r=0, & \rho_t = \sqrt{\rho_0^2+4(d-1)t} \textit{ for all } t\ge 0 \\ \textit{if } r=1, & \rho_t = 2\arccos(e^{-(d-1)t/2}\cos(\rho_0/2)) \textit{ for all } t\ge 0. \end{cases} \tag{5.12}$$

**Remark 11.** *In the case of the sphere $S^2$, the construction in the above theorem matches the one in Theorem 5, but also covers the case of spheres in all dimensions, and has the virtue of being intrinsic.*

*Proof.* We need to compute the expression from (5.6), which is ultimately reduced to the computation of the index forms. Fix two points $x\neq y$ which are not at each other cut-locus.

Take a choice of an orthonormal basis $E_i$, $i=1\dots,d$ at $x$ with $E_1=\dot\gamma(0)$ and use the parallel translation along the geodesic joining $x$ to $y$ to extend it to each point of the geodesic. Then from (2.6) and the formulae following this in Section 4, plus some straightforward computations, reveal that the equation for the distance becomes

$$d\rho_t = -(d-1)\sqrt{r}\tan(\sqrt{r}\rho_t/2)dt, \tag{5.13}$$

which is valid for times $t$ up to the time when $(X_t,Y_t)$ hits the set $C_\epsilon$, for $\epsilon>0$ chosen sufficiently small.

In the case $r=0$, this implies that $\rho_t$ is constant for all times. In the case of $r=-1$, we have that up to the time when $(X_t,Y_t)$ hits $C_\epsilon$,

$$d\rho_t = (d-1)\tanh(\rho_t/2)dt.$$

In the case of the hyperbolic space there is no cut locus, so this equation solves as in (5.11).

In the case of other manifolds of curvature $-1$, we may still have to deal with the fact that the cut-locus is non-trivial, and we reason as follows. First, equation (5.13) shows that $\rho_t$ increases until the time when $(X_t,Y_t)$ enters $C_\epsilon$ ($\epsilon$ small enough). Then the motion moves independently until $(X_t,Y_t)$ leaves $C_{2\epsilon}$, and all this time the distance is certainly greater than half the injectivity radius, and thus is larger than $\rho_0$. Then the processes resume the coupling and the equation above shows that the distance increases again. Overall, in both cases the distance stays greater than $\rho_0$.

In the curvature $r=1$ regime, the equation (5.13) becomes

$$d\rho_t = -(d-1)\tan(\rho_t/2)dt. \tag{5.14}$$

The solution is given by (5.11), at least up to the time when the processes get very close to each other's cut-locus. In the sphere case it is clear that $X_t$ and $Y_t$ never get at each other's cut locus, so we do not have to worry about the cut-locus. In other cases of constant curvature $r=1$, by Mayer's theorem the diameter of $M$ is no greater than $\pi$, so the tangent in the above formula is always non-negative, and thus the distance is decreasing and so we do not have to worry about getting close to the cut-locus. Then the equality $d\rho_t=-(d-1)\tan(\rho_t/2)dt$ implies that $\rho_t\le Ce^{-(d-1)t/2}$ and also that $\rho_t\neq 0$.

Now we move one to the second choice of $O_{x,y}$ from (5.8). The changes are that the index form is computed (again from (2.6)) as

$$\mathcal{I}(J_i,J_i)=\begin{cases} 2\sqrt{r}\cot(\sqrt{r}\rho_t/2), & r\neq 0 \\ \frac{4}{\rho_t}, & r=0 \end{cases},$$

which give in turn the equation for $\rho_t$ as

$$d\rho_t=\begin{cases} (d-1)\sqrt{r}\cot(\sqrt{r}\rho_t/2)dt, & r\neq 0 \\ \frac{2(d-1)}{\rho_t}dt, & r=0, \end{cases}$$

up to the time when $(X_t,Y_t)$ get close to the cut-locus.

When $r=0$, in the particular Euclidean case we get

$$\rho_t=\sqrt{\rho_0^2+4(d-1)t}.$$

In the case when there is a cut locus and the points $X_t$, $Y_t$ get at each other's cut locus, we can argue as before that the distance $\rho_t$ increases, and then eventually we have to switch to the independent Brownian motions inside $C_{2\epsilon}$, and then resume the coupling. At any rate, as soon as the distance becomes less than $i(M)$, $\rho_t$ starts increasing again, so it will never be less than the initial value $\rho_0$.

In the case of negative curvature $r = -1$, we obtain

$$d\rho_t = (d-1)\coth(\rho_t/2)dt,$$

and in the hyperbolic case the solution is given in (5.12). If the cut locus is not empty, then we argue as we already did for the flat case.

Finally, for $r = 1$ the equation is

$$d\rho_t = (d-1)\cot(\rho_t/2)dt,$$

and the solution is given by (5.12) in the case of the sphere, and in the general case we can argue as in the flat case. □

## 6. Shy and Fixed-Distance Couplings on Riemannian Manifolds

In this section we prove a general result about the existence of shy coupling on Riemannian manifolds. Before we launch into various technical details, we state the main result of this section.

**Theorem 12.** *Let $M$ be a complete $d$-dimensional Riemannian manifold with positive injectivity radius and such that for some real number $k$:*

$$k \le Ric_x \text{ for all } x \in M \text{ and } \sup_{x\in M} K_x < \infty, \tag{6.1}$$

*where $Ric$ is the Ricci tensor and $K_x$ stands for the maximum of the sectional curvatures at $x \in M$.*

1. *For $k < 0$, there exists $\epsilon, \delta > 0$ such that for any points $x_0, y_0 \in M$ with $d(x_0, y_0) < \epsilon$ we can find a Markovian coupling of Brownian motions $X_t, Y_t$ starting at $x_0, y_0$ such that $d(X_t, Y_t) \ge d(x_0, y_0)$ for all $t \ge 0$ and $d(X_t, Y_t) = e^{-kt/2} d(x_0, y_0)$ for $0 \le t \le \delta$.*
2. *If $k \ge 0$, then there exists $\epsilon > 0$ such that for any $x_0, y_0 \in M$ with $d(x_0, y_0) < \epsilon$, there exists a Markovian coupling of Brownian motions $X_t, Y_t$ starting at $x_0, y_0$ with $d(X_t, Y_t) = d(x_0, y_0)$ for all $t \ge 0$.*
3. *Moreover, if $k > 0$, then there exists $\epsilon > 0$ such that for any $x_0, y_0 \in M$ with $d(x_0, y_0) < \epsilon$, there exists a Markovian coupling of Brownian motions $X_t, Y_t$ starting at $x_0, y_0$ with $d(X_t, Y_t) = d(x_0, y_0)e^{-kt/2}$ for all $t \ge 0$.*

The plan of the proof is as follows. First we set up an extension of the orthonormal frame bundle (which will be used in the case of even dimensional manifolds). Then we define the equation of the coupling at the level of this frame bundle and we seek a local solution. Once we show the local existence of the coupling, we use patching in order to prove the global existence of the coupling.

We split the proof into several subsections.

6.1. **$N$-frames and the associated bundle.** One of the constructions of the Brownian motion on a $d$-dimensional Riemannian manifold uses the notion of orthonormal frame bundle. We first extend this notion by introducing the following.

**Definition 13.** *Let $N \ge d$ be an integer number. An $N$-frame $U$ in $T_xM$ is a map $U : \mathbb{R}^N \to T_xM$ such that $UU' = Id$. Alternatively, $U$ is an $N$-frame at $T_xM$ if the map $U'$ is an isometric imbedding of $T_xM$ into $\mathbb{R}^N$.*

Using an abuse of language we often say that $U$ is an $N$-frame at $x$ rather than in $T_xM$.

Another way of describing $U$ is via the vectors $X_i = Ue_i$, $i = 1 \dots N$, where $e_i$ are the standard basis vectors in $\mathbb{R}^N$. The condition that $U$ is an $N$-frame is actually equivalent to the condition that

$$\sum_{i=1}^N \langle \xi, X_i\rangle X_i = \xi \text{ for all } \xi \in T_xM. \tag{6.2}$$

Indeed, if $X_i = Ue_i$, then

$$\sum_{i=1}^N \langle \xi, X_i\rangle X_i = U\sum_{i=1}^N \langle U'\xi, e_i\rangle e_i = UU'\xi = \xi \text{ for all } \xi \in T_xM.$$

Conversely, the condition (6.2) determines an $N$-frame $U:\mathbb{R}^N\to T_xM$ by prescribing

$$U\eta=\sum_{i=1}^N\langle \eta,e_i\rangle X_i,$$

noting that

$$U'\xi=\sum_{i=1}^N\langle \xi,X_i\rangle e_i,$$

which under (6.2) gives $UU'=Id$, as needed.

Hence we have different characterizations of an $N$-frame, as a projection, as an isometric embedding and as a set of vectors $Ue_i$.

Suppose we have two points $x,y\in M$, an $N$-frame $\{X_i\}_{i=1}^N$ at $x$, and an isometry $A:T_xM\to T_yM$. Then $\{AX_i\}_{i=1}^N$ is certainly an $N$-frame at $y$ because

$$\sum_{i=1}^N\langle \xi,AX_i\rangle AX_i=A\sum_{i=1}^N\langle A'\xi,X_i\rangle X_i=AA'\xi=\xi.$$

Also, it is easy to see that if $O$ is an orthogonal transformation of $\mathbb{R}^N$ and $U$ is an $N$-frame, then $UO$ is also an $N$-frame. This in fact defines the action of $O(N)$ on the fiber $\mathcal{O}(M)_x$, which is the set of all $N$-frames at $x$. Now we can construct the bundle of $N$-frames denoted by $\mathcal{O}^N(M)$, and for simplicity we will drop the superscript $N$. It is clear that $\mathcal{O}(M)$ is a smooth bundle over $M$ and $\pi:\mathcal{O}(M)\to M$ which assigns to each $N$-frame $U$ in $T_xM$ its base point $x$ (i.e. $\pi U=x$) is a smooth map.

In the terminology of differential geometry, $\mathcal{O}(M)$ is actually a fiber bundle with the fiber being the Stiefel manifold $V_{d,N}$ constructed from the trivial principal bundle $M\times O(N)$ over $M$.

For each fixed $U$ at $T_xM$, the tangent space $T_U\mathcal{O}(M)$ splits into the horizontal part $T_U^H\mathcal{O}(M)$ obtained by lifting tangent vectors from $T_xM$ and the vertical part $T_U^V\mathcal{O}(M)$ which contains a special class of tangent vectors obtained by differentiating curves which are determined by the action of $O(N)$ in the fiber. For references the reader can consult [12] or [25] (the discussion there is intended for the orthonormal frame bundle, but nevertheless most of it extends naturally to this context).

Now, we define the fundamental vector fields $H_i$ on $\mathcal{O}(M)$ by the prescription that at each $U$, $(H_i)_U$ is the lift of the vector $Ue_i$ from $T_{\pi U}M$. The main property here is that the associated Bochner Laplacian

$$\Delta_B=\sum_{i=1}^N H_i^2$$

projects down onto $M$ as the Laplace operator. The proof is as in [25, Section 8.1.3], and for simplicity we just point out the main difference. For a vector $\xi\in\mathbb{R}^N$, let $(H_\xi)_U$ be the horizontal lift of $U\xi$ at $U$. Then with the same proof as [25, Equation 8.30], for any smooth function $f$ on $M$ we have

$$(H_\xi)_U\circ H_\eta(f\circ\pi)=\langle(\mathrm{Hess}f)_{\pi U}U\xi,U\eta\rangle,$$

where $\mathrm{Hess}f$ is the Hessian of $f$ on $M$. Once this is established, we can continue with

$$\begin{aligned}\sum_{i=1}^N(H_i)_UH_i(f\circ\pi)&=\sum_{i=1}^N\langle(\mathrm{Hess}f)_{\pi U}Ue_i,Ue_i\rangle=\sum_{i=1}^N\langle U'(\mathrm{Hess}f)_{\pi U}Ue_i,e_i\rangle\\&=\mathrm{tr}(U'(\mathrm{Hess}f)_{\pi U}U)=\mathrm{tr}((\mathrm{Hess}f)_{\pi U}U\,U')=\mathrm{tr}((\mathrm{Hess}f)_{\pi U})\\&=(\Delta_Mf)(\pi U),\end{aligned}$$

where we used that the Laplacian on $M$ is simply the trace of the Hessian. Thus

$$\pi_*\Delta_B=\Delta_M. \tag{6.3}$$

Under the assumptions in (6.1), the Ricci curvature is bounded from below from which we learn that the Brownian motion on $M$ does not explode. Thus the Brownian motion constructed on $\mathcal{O}(M)$ (more appropriately the solution to the martingale problem for $\Delta_B$) projects down into the Brownian motion on $M$ and exists for all times.

6.2. **The Coupling SDE.** Now we want to couple Brownian motions on $M$, and for this matter we consider couplings of a similar form to the one in the previous section. More precisely, for given points $x_0, y_0 \in M$ and $N$-frames $U_0$ at $x_0$ and $V_0$ at $y_0$, consider the system

$$\begin{cases} dU_t = \sum_{i=1}^N H_i(U_t) \circ dW_t^i \\ dV_t = \sum_{i=1}^N H_i(V_t) \circ dB_t^i \\ dB_t = O_{U_t,V_t} dW_t \\ X_t = \pi U_t \\ Y_t = \pi V_t. \end{cases} \tag{6.4}$$

Here $W_t$ is an $N$-dimensional Brownian motion and $O_{U,V}$ is an orthogonal $N \times N$ matrix which depends smoothly on $U, V$, at least on a subset of $\mathcal{O}(M) \times \mathcal{O}(M)$ which will be specified later on. This insures that $B_t$ is also an $N$-dimensional Brownian motion. We do not impose additional conditions on the matrix $O_{U_t,V_t}$ yet.

The same arguments as in [12, Section 6.5] show that the generator of the diffusion $(U_t, V_t)$ is given by

$$\Delta^c = \Delta_{B,1} + \Delta_{B,2} + 2\sum_{i=1}^N H_{e_i^*,2} H_{i,1}$$

where the subscript 1 or 2 represents the action with respect to the first or the second variable, and $e_i^* = O_{U,V} e_i$.

Let $\rho_t = d(X_t, Y_t)$ be the distance function between the processes $X_t$ and $Y_t$ and let $\widetilde{\rho}(t) = d(\pi U_t, \pi V_t)$. Also let $\widetilde{d}(U, V) = d(\pi U, \pi V)$ be the lift of the distance function from $M$ into $\mathcal{O}(M)$. Using Itô's formula we have that

$$d\widetilde{\rho}_t = \left((H_{i,1} + H_{e_i^*,2})\widetilde{d}\right)(U_t, V_t) dW_t + \frac{1}{2}\left(\Delta^c \widetilde{d}\right)(U_t, V_t) dt, \tag{6.5}$$

which is certainly valid in the region where $\pi U_t$ and $\pi V_t$ are not at each other's cut-locus.

The first variation formula gives

$$(H_{i,1} + H_{e_i^*,2})\widetilde{d}(U, V) = \langle U e_i, \gamma_{X,Y} \rangle_{\pi V} - \langle V O_{U,V} e_i, \gamma_{X,Y} \rangle_{\pi U},$$

where $X = \pi U$, $Y = \pi V$, and $\gamma_{X,Y}$ is the minimizing geodesic joining $X$ to $Y$, run at unit speed. The bounded variation part comes from the second variation formula and produces

$$(\Delta^c \widetilde{d})(U, V) = \sum_{i=1}^N \mathcal{I}(J_i, J_i), \tag{6.6}$$

where $J_i$ is the Jacobi field along the geodesic joining $\pi U$ to $\pi V$, with values $U e_i$, $V O_{U,V} e_i$ at the endpoints.

In order to cancel the martingale part from (6.5), we need to impose the condition

$$\langle U e_i, \dot{\gamma}_{X,Y} \rangle_{\pi V} - \langle V O_{U,V} e_i, \dot{\gamma}_{X,Y} \rangle_{\pi U} = 0.$$

6.3. **Local Construction.** This part of the proof consists in showing that there exists $\eta > 0$ sufficiently small such that for any $x, y \in M$ with $d(x, y) < \eta$ there is a smooth choice of $O_{U,V}$ on $\mathcal{N}_\eta(x, y) = \pi^{-1}(B(x, \eta)) \times \pi^{-1}(B(y, \eta))$ for which

$$\langle U e_i, \dot{\gamma}_{\pi U, \pi V} \rangle_{\pi U} - \langle V O_{U,V} e_i, \dot{\gamma}_{\pi U, \pi V} \rangle_{\pi V} = 0 \text{ for } (U, V) \in \mathcal{N}_\eta(x, y) \tag{6.7}$$

and

$$\sum_{i=1}^N \mathcal{I}(J_i, J_i) = -k d(x, y), \text{ for } (U, V) \in \mathcal{N}_\eta(x, y), \tag{6.8}$$

where $J_i$ are the Jacobi fields with boundary values $U e_i$ and $V O_{U,V} e_i$ at the endpoints of the geodesic joining $\pi U$ and $\pi V$. Note here that for small $\eta$, there is a unique geodesic joining $\pi U$ and $\pi V$, so everything is well defined in this case.

Take $\eta < i(M)/3$, where $i(M)$ is the injectivity radius of $M$. In fact we are going to choose possibly smaller values of $\eta$ later in the construction, but for now assume that it is smaller than $i(M)/3$.

Now, assume that $x_0, y_0 \in M$ are two fixed starting points with distance $d(x_0, y_0) < \eta$. We will construct the coupling $(U_t, V_t)$ in $\mathcal{N}_\eta(x_0, y_0)$.

We can choose an orthonormal basis $E_1, E_2, \dots, E_d$ at $x$ such that $E_1 = \dot\gamma_{x,y}(0)$ and such that each $E_j$ depends smoothly on $(x, y) \in B(x_0, \eta) \times B(y_0, \eta)$. We can extend this basis $E_1, \dots, E_d$ along $\gamma_{x,y}$ and continue to call it $E_1, \dots, E_d$. Now, condition (6.7) becomes

$$U'\dot\gamma_{x,y} = O'_{U,V}V'\dot\gamma_{x,y}. \tag{6.9}$$

Next, let us denote $J_{1,j}$ the Jacobi field along the minimizing geodesic joining $\pi U$ to $\pi V$ such that it equals $E_j$ at $\pi U$ and $0$ at $\pi V$. Similarly let $J_{2,j}$ be the Jacobi field which is $0$ at $\pi U$ and $E_j$ at $\pi V$. Then, since

$$J_i = \sum_{j=1}^{d} \langle Ue_i, E_j\rangle J_{1,j} + \sum_{j=1}^{d} \langle VO_{U,V}e_i, E_j\rangle J_{2,j}$$

it follows that

$$\sum_{i=1}^{N} \mathcal{I}(J_i, J_i) = \sum_{j=2}^{d} \mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d} \mathcal{I}(J_{2,j}, J_{2,j}) + 2\sum_{j,k=2}^{d} \langle O'_{U,V}V'E_j, U'E_k\rangle \mathcal{I}(J_{1,j}, J_{2,k}). \tag{6.10}$$

The expression given by the last sum can be simplified as follows. Let $\tau_{x,y}$ stand for the parallel transport map from $T_xM$ to $T_yM$ along the minimizing geodesic $\gamma_{x,y}$. Consider the bilinear map $\Lambda_{x,y} : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\Lambda_{x,y}(\xi, \eta) = \mathcal{I}(J_{1,\xi}, J_{2,\eta}),$$

where $J_{1,\xi}$ is the Jacobi field along $\gamma_{x,y}$ which is $\xi$ at $x$ and $0$ at $y$, and $J_{2,\eta}$ is $0$ at $x$ and $\tau_{x,y}\eta$ at $y$. Another way of looking at this is as a linear map from $T_xM$ into itself, map which we still call $\Lambda_{x,y}$. We can see this map also as a linear transformation preserving the orthogonal to $\dot\gamma_{x,y}$ at $x$ and we will denote this restriction also by $\Lambda_{x,y}$. In fact the actions of $\Lambda_{x,y}$ and its transpose on $\dot\gamma_{x,y}$ are zero.

With this notation, it is not hard to see that for $N$-frames $U$ and $V$ at $x$, respectively at $y$, we have

$$\sum_{j,k=2}^{d} \langle O'_{U,V}V'E_j, U'E_k\rangle \mathcal{I}(J_{1,j}, J_{2,k}) = \mathrm{tr}(UO'_{U,V}V'\tau_{x,y}\Lambda_{x,y}). \tag{6.11}$$

For the first part of the theorem we want to find a map $O_{U,V}$ such that the quantity in (6.10) equals $-kd(x, y) =$. For simplicity of notations, we are going to denote $d(x, y) = \rho$.

To carry this task through, we are going to use the following standard comparison result, whose proof can be found for instance in [6, pp. 216-217].

**Lemma 14.** *Assume that $M$ and $\widetilde{M}$ are two manifolds and $\gamma$, $\widetilde\gamma$ are two normalized geodesics defined on $[0, \rho]$ such that $\widetilde\gamma$ does not have conjugate points. Assume that $J_t$ and $\widetilde{J}_t$ are two Jacobi vector fields along $\gamma$, respectively $\widetilde\gamma$, such that $J_0 = \widetilde{J}_0 = 0$, $|J_\rho| = |\widetilde{J}_\rho|$, and*

$$K^+(\gamma(t)) \le \widetilde{K}^-(\widetilde\gamma(t)),$$

*where $K^+(x)$ is the maximum of the sectional curvature at $x$ and $\widetilde{K}^-(\widetilde{x})$ is the minimum of the sectional curvature at $\widetilde{x}$. Then we have*

$$\mathcal{I}(\widetilde{J}, \widetilde{J}) \le \mathcal{I}(J, J). \tag{6.12}$$

Since the sectional curvature is bounded from above, $K_x \le 1/r^2$ for all $x \in M$ for a small enough $r$, and for reasons which will become apparent immediately we take $r < (1/4)^{1/3}$. In turn, this shows that the injectivity radius is at least $r\pi$ (see [6, p. 218]).

With this, for points $x, y \in M$ at distance $\rho = d(x, y) < \pi r$, comparing the index form of the manifold $M$ with the index form of a sphere of radius $r$, for geodesics of length $\rho < \pi r$, we obtain

$$\mathcal{I}(\widetilde{J}, \widetilde{J}) \le \mathcal{I}(J, J),$$

where $J, \widetilde{J}$ are as in the Lemma 14. On the other hand, for the $d$-dimensional sphere $S^d$ we have $\widetilde{J}(s) = w_2(s)\widetilde{E}(s)$, where $w_2$ is given by (2.6) and $\widetilde{E}$ is the parallel transport of $\widetilde{E}_0 \in T_{\widetilde{\gamma}(0)}S^d$ along $\widetilde{\gamma}$. From (2.5) we conclude that

$$\mathcal{I}(\widetilde{J}, \widetilde{J}) = \dot{w}_2(\rho) = \sqrt{r}\cot(\sqrt{r}\rho)$$

and consequently, because we choose $r < (1/4)^{1/3}$, we obtain

$$0 < \sqrt{r}\cot(\sqrt{r}\rho) = \mathcal{I}(\widetilde{J}, \widetilde{J}) \leq \mathcal{I}(J, J). \tag{6.13}$$

We now choose $\eta$ sufficiently small, for instance smaller than $r$ above, and therefore also less than a third of the injectivity radius of $M$.

Now we want to choose $O_{U,V}$ so that

$$U'\dot{\gamma}_{x,y} = O'_{U,V}V'\dot{\gamma}_{x,y}$$

and

$$\operatorname{tr}(UO'_{U,V}V'\tau_{x,y}\Lambda_{x,y}) = -\frac{1}{2}\left(\sum_{j=2}^{d}\mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d}\mathcal{I}(J_{2,j}, J_{2,j}) - k\rho\right). \tag{6.14}$$

To show this, we recall another standard result in Riemannian geometry.

**Lemma 15.** *Assume $\gamma$ is a normalized geodesic on $[0, \rho]$ without conjugate points on it. If $J$ and $V$ are two vector fields with the same boundary values, and $J$ is also a Jacobi field, then*

$$\mathcal{I}(J, J) \leq \mathcal{I}(V, V). \tag{6.15}$$

This result can be easily deduced from [6, Lemma 2.2, Chapter 10] which is the above statements for the case both $J$ and $V$ cancel at one end of the geodesic. However, if $J$ and $V$ have the same values at the endpoints, then due to the equation of the Jacobi field and an integration by parts, we have

$$\mathcal{I}(J, V) = \langle \dot{J}(\rho), V(\rho)\rangle - \langle \dot{J}(0), V(0)\rangle = \langle \dot{J}(\rho), J(\rho)\rangle - \langle \dot{J}(0), J(0)\rangle = \mathcal{I}(J, J),$$

so using the index lemma for the vector field $V - J$ and the trivial Jacobi field $0$, we obtain

$$0 \leq \mathcal{I}(V - J, V - J) = \mathcal{I}(V, V) - 2\mathcal{I}(V, J) + \mathcal{I}(J, J) = \mathcal{I}(V, V) - \mathcal{I}(J, J).$$

Now,

$$\sum_{j=2}^{d}\mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d}\mathcal{I}(J_{2,j}, J_{2,j}) + 2\sum_{j=2}^{d}\mathcal{I}(J_{1,j}, J_{2,j}) = \sum_{j=2}^{d}\mathcal{I}(J_{1,j} + J_{2,j}, J_{1,j} + J_{2,j}).$$

On the other hand, using the above comparison theorem with the vectors $E_j$ in place of $V$ and $J_{1,j} + J_{2,j}$ as the Jacobi field $J$, we obtain

$$\begin{aligned}\sum_{j=2}^{d}\mathcal{I}(J_{1,j} + J_{2,j}, J_{1,j} + J_{2,j}) \leq \sum_{j=2}^{d}\mathcal{I}(E_j, E_j) &= \sum_{j=2}^{d}\int_0^\rho \left(|\dot{E}_j(s)|^2 - \langle R(\dot{\gamma}(s), E_j(s))E_j(s), \dot{\gamma}(s)\rangle\right) ds \\ &= -\int_0^\rho Ric_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))ds \leq -k\rho\end{aligned}$$

where $\rho = d(x, y)$, and therefore

$$2\sum_{j=2}^{d}\mathcal{I}(J_{1,j}, J_{2,j}) \leq -(\sum_{j=2}^{d}\mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d}\mathcal{I}(J_{2,j}, J_{2,j})) - k\rho. \tag{6.16}$$

In the basis $E_1 = \dot{\gamma}_{x,y}, E_2, \ldots, E_d$ we can take

$$f_j = U'E_j \quad \text{and} \quad h_j = V'E_j, \qquad j = 1, \ldots, d.$$

To choose the matrix $\Delta_{x,y}$ as in (6.14) we treat separately the cases of odd and even dimensional manifolds, as follows.

*Case I: $d$ is odd.* In this case we take $N = d$, so we are back to the classical situation of the orthonormal frame bundle. Let $A_U$ and $A_V$ be the (unique) orthogonal matrices which send $e_j$ into $f_j$, respectively $e_j$ into $h_j$, $j = 1, \dots, d$. We will choose the matrix $O_{U,V}$ such that, in addition to (6.14) we also have

$$A'_U O'_{U,V} A_V e_1 = e_1.$$

This is done as follows. Pick an orthogonal matrix $B_{x,y}$ such that

$$B_{x,y} e_1 = e_1 \text{ and } \operatorname{tr}(B_{x,y}\Delta_{x,y}) = 0,$$

and with this we take

$$O_{U,V} = A_V B'_{x,y} A'_U.$$

To get to terms with the matrix $B_{x,y}$, we choose it to be given in matrix form by

$$B_{x,y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}. \tag{6.17}$$

This is where we actually use the fact that the dimension $d$ is odd: in the above representation we use on the diagonal $(d-1)/2$ blocks of $2\times 2$ unitary matrices. With this choice, we clearly have $B_{x,y}e_1 = e_1$, and it remains to pick $\alpha \in [0, 2\pi]$ such that

$$2\cos(\alpha)\operatorname{tr}(\Delta_{x,y}) + 2\sin(\alpha)F_{x,y} = -\left(\sum_{j=2}^{d} \mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d} \mathcal{I}(J_{2,j}, J_{2,j})\right) - k\rho,$$

where $F_{x,y} = \sum_{i=2}^{d-1}\langle \Delta_{x,y}e_i, e_{i+1}\rangle - \sum_{i=3}^{d}\langle \Delta_{x,y}e_i, e_{i-1}\rangle$. The key is that (6.16) is nothing but the statement that

$$2\operatorname{tr}(\Delta_{x,y}) \le -\left(\sum_{j=2}^{d} \mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d} \mathcal{I}(J_{2,j}, J_{2,j})\right) - k\rho.$$

Now, inequality (6.13) gives that for any fixed real number $k$,

$$-\left(\sum_{j=2}^{d} \mathcal{I}(J_{1,j}, J_{1,j}) + \sum_{j=2}^{d} \mathcal{I}(J_{2,j}, J_{2,j})\right) - k\rho < -2\sqrt{r}\cot(\sqrt{r}\rho) - k\rho < 0$$

for small enough $\rho$. Finally, simple trigonometry shows that for any $a < c < 0$ and any $b$, the equation

$$\cos(\alpha)a + \sin(\alpha)b = c$$

has the solution

$$\alpha = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) + \arccos\left(\frac{c}{\sqrt{a^2+b^2}}\right).$$

*Case II: $d$ is even.* In this case we use $N = d+1$. The difference from the previous case is that this time we consider the matrix $A_U$ which sends $e_j$ into $f_j$, $j = 1, \dots, d$, and the vector $e_{d+1}$ into the exterior product of $f_1, \dots, f_d$, and we define the matrix $A_V$ in a similar fashion. Note that with this choice, $A_U$ and $A_V$ are orthogonal matrices. The rest of the argument is now the same argument as in the case when $d$ is odd, with the choice of $B_{U,V}$ as a $(d+1)\times(d+1)$ matrix such as the one in (6.17).

Let's wrap up the main findings of this subsection. We showed that there exists a matrix $O_{U,V}$ which depends smoothly on $(U,V) \in \mathcal{N}_\eta(x_0, y_0)$ such that (6.7) and (6.8) are satisfied. In fact we proved that for small enough $\eta > 0$, as long as the distance between $x_0$ and $y_0$ is less than $\eta/2$ and the process $(X_t, Y_t)$ stays inside $B(x_0,\eta)\times B(y_0,\eta)$, the distance function satisfies

$$\rho_t = e^{-kt/2}\rho_0, \tag{6.18}$$

6.4. **The construction of the coupling.** Consider first two independent $N$-dimensional Brownian motions $W_t$ and $\widetilde{W}_t$. For a given stopping time $\tau$, we denote $W_{t,\tau} = W_t - W_\tau$.

We have proved that for a small enough $\eta > 0$ and any $x, y$ with $d(x,y) < \eta$ there exists a smooth choice $O_{U,V}$ on $\mathcal{N}_\eta(x,y)$. We will now use this to give a construction of the coupling as indicated in the statement of the theorem.

For any $\eta > 0$ we define the $\eta$-neighborhood of the diagonal in $M \times M$ by

$$D_\eta = \{(x,y) : d(x,y) \le \eta\},$$

and we also set

$$\mathcal{D}_\eta = \{(U,V) \in \mathcal{O}(M) \times \mathcal{O}(M) : (\pi U, \pi V) \in D_\eta\}.$$

For a fixed pair of points $(x_0, y_0) \in D_{\eta/4}$ and frames $U_0, V_0$ at $x_0$, respectively at $y_0$, we consider an orthonormal basis $E_1, \dots E_d$ at $x_0$ with $E_1 = \dot\gamma_{x_0,y_0}(0)$ and extend this to a local orthonormal basis on $B(x_0, 2\eta)$ and then by parallel transport also to $B(y_0, 2\eta)$. Using the local recipe outlined above we can construct a coupling with $\rho_t = e^{-kt/2}\rho_0$ up to the first time $t$ when the base process $(X_t, Y_t)$ hits the boundary of the set $B(x_0,\eta) \times B(y_0,\eta)$. Let's call this exit time $\tau_1$. At $(x_1, y_1) = (X_{\tau_1}, Y_{\tau_1})$ we have the orthogonal basis $E_1, \dots, E_d$ used in the local construction, which at $x_1$ satisfies $E_1 = \dot\gamma_{x_1,y_1}$, and $U_1 := U_{\tau_1}$ and $V_1 := V_{\tau_1}$ are the frames obtained from (6.4).

The next step is to extend the construction of the coupling beyond time $\tau_1$. There are two cases to be considered here.

If the point $(x_1, y_1)$ lies inside $D_{\eta/2}$, we can use the starting point $(x_1, y_1)$ and continue to run $(U_t, V_t)$ following (6.4) using now the Brownian motion $W_{t,\tau_1}$ with the time range $t \ge \tau_1$. As above we let $\tau_2$ be the first time the process $(X_{t+\tau_1}, Y_{t+\tau_1})$ hits the boundary of $B(x_1,\eta) \times B(y_1,\eta)$, and we set $(x_2, y_2) = (X_{\tau_1+\tau_2}, Y_{\tau_1+\tau_2})$ and also $U_2 = U_{\tau_1+\tau_2}$ and $V_2 = V_{\tau_1+\tau_2}$.

On the other hand, if the point $(x_1, y_1)$ lands outside $D_{\eta/2}$, then we run the motions $U_t$ and $V_t$ for $t \ge \tau_1$ with the system

$$\begin{cases} dU_t = \sum_{i=1}^N H_i(U_t) \circ dW^i_{t,\tau_1} \\ dV_t = \sum_{i=1}^N H_i(V_t) \circ d\widetilde{W}^i_{t,\tau_1} \\ X_t = \pi U_t \\ Y_t = \pi V_t. \end{cases}$$

In other words, $U_t, V_t$ run as independent Brownian motions on $\mathcal{O}(M) \times \mathcal{O}(M)$, and $X_t, Y_t$ run as independent Brownian motions on the base manifold $M$. We continue with this construction for time $t$ in the interval $[\tau_1, \tau_1 + \tau_2]$, where the terminal time $\tau_1 + \tau_2$ is the first time the process $(X_t, Y_t)$ lands in $D_{\eta/4}$, and we denote $(x_2, y_2) = (X_{\tau_1+\tau_2}, Y_{\tau_1+\tau_2})$.

In both cases above we constructed the processes $U_t, V_t$ defined up to the time $\tau_1 + \tau_2$, and $(x_2, y_2)$ is either in $D_{\eta/2}$ or outside it. Inductively, we can now repeat the construction above, to show that we can extend the construction of the processes for another $\tau_3$ units of time, and so on. If for a certain $n$, $\tau_n = +\infty$, then we certainly take all other stopping times $\tau_m = 0$ for $m > n$.

One of the main problems is to show that the construction can be extended for all times $t \ge 0$, in other words that

$$\sum_{n \ge 1} \tau_n = +\infty.$$

We are going to do this separately for the first part of the theorem, and argue differently for the second and third part.

In the first case, where $k < 0$, the idea is that as long as the process $(X_t, Y_t)$ stays inside $D_{\eta/2}$, from (6.18) we have that the distance process $\rho_t$ satisfies

$$\rho_t = e^{-kt/2}\rho_0,$$

thus increasing. This means that if $\eta$ is small enough, then in finite (deterministic) time the process $(X_t, Y_t)$ exits $D_{\eta/2}$. Once the process $(X_t, Y_t)$ exits the set $D_{\eta/2}$, $X_t$ and $Y_t$ run independently until they hit the set $D_{\eta/4}$, and then they stay in $D_{\eta/2}$ for at most a finite (deterministic) amount of time, after which they exit again $D_{\eta/2}$. In particular we see that the processes $X_t, Y_t$ have to run independently infinitely many times, and it is this fact that allows us to show that $\sum_{n\geq 1}\tau_n = +\infty$. This is done using the Borel-Cantelli's lemma.

For the moment assume that we have two independent Brownian motions $X_t, Y_t$ starting at $x_0, y_0$ with $d(x_0, y_0) = \eta/2$. If $\tau$ is the first time when $X_t, Y_t$ are within distance $\eta/4$ to each other, we want to get an estimate on $\mathbb{P}(\tau > \delta)$ for some $\delta > 0$. To do this, we use the following inclusion

$$\{\zeta_{X,\eta/16} > \delta\} \cap \{\zeta_{Y,\eta/16} > \delta\} \subset \{\tau > \delta\}$$

where $\zeta_{X,\eta/16}$ is the first exit time of $X_t$ from the ball $B(x_0, \eta/16)$ and similarly $\zeta_{Y,\eta/16}$ is the first time $Y_t$ exits the ball $B(y_0, \eta/16)$. This inclusion can be stated in words as follows. If $X_t$ and $Y_t$ stay inside $B(x_0, \eta/16)$, respectively $B(y_0, \eta/16)$, up to time $\delta$, and since $x_0, y_0$ are distance $\eta/2$ apart, it follows that $X_t$ and $Y_t$ are not within $\eta/4$ of each other in the time interval $[0, \delta]$. The conclusion we draw from this is that

$$\mathbb{P}(\tau > \delta) \geq \mathbb{P}(\zeta_{X,\eta/16} > \delta)\mathbb{P}(\zeta_{Y,\eta/16} > \delta).$$

Finally, since the the Ricci curvature is bounded below, we can invoke now the estimate on the exit times from balls, for instance [12, Theorem 3.6.1], to obtain that for any point $x$ on $M$ we have

$$\mathbb{P}_x(\zeta_{\eta/16} \leq \delta) \leq e^{-Cr^2/\delta},$$

where the constant $C > 0$ depends only on the lower bound on the Ricci curvature and the dimension of the manifold. Thus for a fixed $\eta > 0$ we obtain that

$$\mathbb{P}_x(\zeta_{\eta/16} > \delta) > 1 - e^{-C\eta^2/\delta} := C_2 > 0, \tag{6.19}$$

for a certain constant $C > 0$, and therefore

$$\mathbb{P}(\tau > \delta) \geq C_2^2.$$

With this at hand we get that

$$\sum_{n\geq 1}\mathbb{P}(\tau_n > \delta) = +\infty,$$

and using Borel-Cantelli's lemma we conclude that $\sum_{n\geq 1}\tau_n = +\infty$, which shows that the construction of the coupling extends for all times $t \geq 0$.

For the other case of fixed or decreasing distance coupling, the point is complementary to the previous one. More precisely, in the above proof it was the independent motions which played the main role, while here the main role is played by the coupling. To get to terms, note that if we start the coupling with points $x_0, y_0$ such that $d(x_0, y_0) < \eta/4$, then, since the distance between the processes does not increase, the process $(X_t, Y_t)$ stays in $D_{\eta/2}$ up to the time $\sum_{n\geq 1}\tau_n$. The issue is to show that this sum is always infinite. What we want to do is to find a lower bound on $\mathbb{P}(\tau_1 > \delta)$. Using the same notation as above, we have

$$\{\zeta_{X,\eta/16} > \delta\} \subset \{\tau_1 > \delta\}. \tag{6.20}$$

To see this, we follow the construction until either $X$ or $Y$ hit the ball of radius $\eta$ centered at $x_0$, respectively $y_0$. Now, if $X$ stays inside $B(x_0, \eta/16)$ on the time interval $[0, \delta]$, since $d(x_0, y_0) < \eta/4$ and the processes remain at fixed or non-increasing distance, an application of the triangle inequality shows that $Y$ remains inside $B(y_0, 9\delta/16)$ on the time interval $[0, \delta]$, which in turn implies (6.20). Using again (6.19) we get that

$$\mathbb{P}(\tau_1 > \delta) \geq C_3 > 0$$

for a constant $C_3$ which is independent of the starting points. Since this is applicable to all stopping times $\tau_n$, we learn again from Borel-Cantelli's lemma that $\sum_{n\geq 1}\tau_n = +\infty$.

6.5. **Finishing off.** In the previous section we constructed the coupling and we proved that it is defined for all times. We want now to show that the construction actually does what the Theorem asks for. This is already spelled out in the previous subsection in a certain form.

For the first part ($k < 0$), on each of the regions where the coupling is inside $D_{\eta/2}$, due to (6.18), we see that the distance is non-decreasing, and therefore it is larger than the starting distance which is at most $\eta/4$. On the other hand, if the coupling exits $D_{\eta/2}$, then it runs as independent Brownian motions until it hits again $D_{\eta/4}$, and consequently the distance is at least $\eta/4$ apart. In both regimes the distance does not get smaller than the starting distance and this concludes the proof of the first part of the theorem.

For the second part, the process never leaves $D_{\eta/2}$ and for all times it remains at a fixed distance. The last case, when the distance decays exponentially, being similar, we omit it. This concludes the proof of Theorem 12.

## 7. Refinements and Comments

The proof of Theorem 12 spreads on several pages, and some comments on it are in order. The first observation is that the conditions imposed are essential for the construction. For example the positivity of the injectivity radius is needed for the local construction. The Ricci curvature bounded from below insures the non-explosion of the Brownian motion on one hand, and on the other hand it is important in the estimate of the exit times employed in the proof of the global existence of the coupling.

That the scalar curvature is bounded from above does not seem to be optimal even though it is an important piece in the proof of the existence of the coupling via the index form comparison on $M$ with the index form of a sphere. Geometrically, we certainly need to make sure that the Brownian motions we try to couple do not get trapped in regions of extremely high scalar curvature where the Brownian motions tend to get close to one another. It seems though that the optimal condition would be that the injectivity radius of the manifold is positive. However this certainly requires a different argument from the one provided here.

Another aspect is that the global existence of the choice of the map $O_{U,V}$ is tied to the existence of a smooth choice of an orthonormal frame on $M$. On an arbitrary Riemannian manifold this can be done only locally and this is why we had to go one more step, from the local existence of the coupling to its global existence. There are though a few cases when the existence can be proved globally, one of which is the case of surfaces. In this case, for any two points $x, y$ not at each other cut-locus, there is a single perpendicular direction to the geodesic joining $x$ and $y$. Using this we can show that there is a global choice of $O_{U,V}$ as long as $\pi U, \pi V$ are not at each other cut-locus.

Another case in which we can construct a global version of $O_{U,V}$ is the one in which $M$ is parallelizable, namely the tangent bundle is trivializable, or otherwise put, there exist vector fields $X_1, X_2, \dots X_d$ which are independent at each point. This amounts to the existence of a global orthonormal frame bundle. It is for instance the case of $S^3$ and $S^7$ and also of any Lie group with the left or right invariant metric.

The couplings we constructed in Theorem 12 are defined for all times $t \geq 0$, and the conditions in (6.1) were necessary in the proof. There is however a case when the injectivity and upper bound on the sectional condition can be dispensed of if one only needs the coupling to be defined up to the first exit time of the coupling from a relatively compact set. For completeness we record the result here and use it in the next section. The proof is the same as the one given above adjusted with a stopping time.

**Theorem 16.** *Let $M$ be a complete $d$-dimensional Riemannian manifold and $D \subset M$ a relatively compact open set of $M$ with a smooth boundary. Then, there exists $\epsilon > 0$ such that for any $x, y \in D$ with $d(x, y) < \epsilon$, there exist a shy coupling of two Brownian motions on $M$ starting at $x$ and $y$, defined up to the first exit time of either of the processes from $D$.*

*If in addition $Ric \geq 0$, there also exists a fixed-distance coupling Brownian motions on $M$ starting at $x$ and $y$, defined up to the first exit time of either of the processes from $D$.*

The suggestion given by Kendall in [16] for the construction of the shy coupling is to use the same orthogonal transformation as in (5.8) which is a form of perverse coupling (in the terminology of [16]). However, this is not sufficient to get the fixed distance coupling. Particularly this is very clearly illustrated in the case of surfaces. Indeed, since the dimension is 2, we have just one dimension left in the orthogonal to the geodesic joining $X_t$ and $Y_t$ and then there are essentially only two choices of an orthogonal map from $T_x$ to $T_y$ (for $x, y$ not at each other cut-locus) which preserves the geodesic direction. These are the ones in (5.7) and (5.8). At least in the constant curvature case manifolds as we showed in Theorem 10 we do not get a fixed distance coupling.

We want to point out that one can get a shy coupling using stochastic flows. In short, the idea is to impose conditions such that the flow stays a Brownian motion and this can be done if the direction in the Cameron-Martin space satisfies a certain ode. If the initial value of this direction is non-zero everywhere then we obtains weak form of a shy coupling. See for details [12, 13].

At last, it would be very interesting to see a construction of the fixed-distance coupling in the same spirit as the one pointed in Remark 9. This would probably work only in special cases as for example homogeneous manifolds.

## 8. Applications

As an application of the fixed-distance coupling we present a proof of the following maximum principle for the gradient of harmonic functions.

**Theorem 17.** *Let $M$ be a Riemannian manifold with non-negative Ricci curvature and let $u : M \to \mathbb{R}$ be a harmonic function on $M$. Then, for any relatively compact open set $D$ with smooth boundary we have*

$$\max_{x\in\overline{D}} |\nabla u(x)| = \max_{x\in\partial D} |\nabla u(x)|. \tag{8.1}$$

*Proof.* Fix an arbitrary point $x \in D$. Then there is a geodesic $\gamma$ such that $\gamma(0) = x$ and

$$v(x) := |\nabla u(x)| = \lim_{h\to 0} \frac{u(\gamma(h)) - u(x)}{h}.$$

In particular, for a small enough $h > 0$, we can consider the fixed distance coupling from Theorem 16 started at $\gamma(h)$ and $x$, and run it up until the stopping time $\zeta$, defined as the first time when either of the processes $X_t$ or $Y_t$ hit the boundary of $D$. On the other hand, since the function $u$ is harmonic, $u(X_t)$ and $u(Y_t)$ are local martingales, and in fact, since $u$ is bounded on $D$, we can write

$$u(\gamma(h)) - u(x) = \mathbb{E}[u(X_\zeta) - u(Y_\zeta)].$$

The upshot of this equality is that there must be an $\omega$ in the probability space where the processes $X_t$ and $Y_t$ are defined, such that

$$u(\gamma(h)) - u(x) \le u(X_\zeta(\omega)) - u(Y_\zeta(\omega))$$

Since $d(X_\zeta, Y_\zeta) = d(\gamma(h), x) = h$, we can find a point $\xi$ on the geodesic joining $X_\zeta$ with $Y_\zeta$ such that

$$u(\gamma(h)) - u(x) \le u(X_\zeta(\omega)) - u(Y_\zeta(\omega)) \le |\nabla u(\xi)|h.$$

Since either $X_t$ or $Y_t$ are on the boundary of $D$, we conclude that $\xi$ is distance $h$ or less from the boundary $\partial D$.

Thus, as $h$ goes to $0$, from the compactness of $\partial D$, we can find a point $\alpha \in \partial D$ such that

$$v(x) = |\nabla u(x)| \le |\nabla u(\alpha)| \le \max_{x\in\partial D} v(x),$$

which concludes the proof. □

Another application with a similar flavor is to the gradient of a solution to the heat equation.

**Theorem 18.** *Let $M$ be a Riemannian manifold with non-negative Ricci curvature, $D \subset M$ a relatively compact open set with smooth boundary, and $u$ a classical smooth solution to the heat equation $\partial_t u = \frac{1}{2}\Delta u$ on an open relatively compact set $\Omega$ containing the closure of $D$. For $x \in D$, let $\mu_{x,D}$ be the distribution of the hitting time $\zeta$ of $\partial D$ for a Brownian motion started at $x$. Then, for each $x \in D$ and $0 < s < t$*

$$|\nabla u_t(x)| \le \int_0^s |\nabla u_{t-\sigma}|_{\partial D}\mu_{x,D}(d\sigma) + \mu_{x,D}((s,\infty))|\nabla u_{t-s}|_D, \tag{8.2}$$

*where for a bounded set $K$ and a continuous function $w$ on $K$, we denoted $|w|_K = \sup_{x\in K} |w(x)|$.*

*Note that in the case when $u$ is harmonic, (8.2) implies (8.1).*

*Proof.* As in the previous proof, consider a point $x$ and let $\gamma$ be a geodesic such that $\gamma(0) = x$ and

$$|\nabla u_t(x)| = \lim_{h\to 0} \frac{u_t(\gamma(h)) - u_t(x)}{h}$$

Using the fixed distance coupling started at $\gamma(h)$ and $x$ given by Theorem 12, and with $\zeta$ the first time either $X_t$ or $Y_t$ hit $\partial D$, and combinining with the fact that $u_{t-s}(X_s)$ and $u_{t-s}(Y_s)$ are local martingales, we get for $h > 0$,

$$\begin{aligned} u_t(\gamma(h)) - u_t(x) &= \mathbb{E}[u_{t-s\wedge\zeta}(X_{s\wedge\zeta}) - u_{t-s\wedge\zeta}(Y_{s\wedge\zeta})] \\ &= \mathbb{E}[u_{t-\zeta}(X_\zeta) - u_{t-\zeta}(Y_\zeta), \zeta \le s] + \mathbb{E}[u_{t-s}(X_s) - u_{t-s}(Y_s), s < \zeta] \\ &\le h\mathbb{E}[|\nabla u_{t-\zeta}|_{(\partial D)_h}, \zeta \le s] + h\mathbb{E}[|\nabla u_{t-s}|_D, s < \zeta] \end{aligned}$$

where $K_h$ denotes the set of points of $M$ at distance at most $h$ from $K$. Dividing by $h$ and letting $h$ go to $0$, the first part follows.

For the second part, if $u$ is a harmonic function and the maximum of $u$ on $\overline{D}$ is attained at a point $x \in D$, then taking this point in (8.2) gives (8.1) □

8.1. **The Brownian Lion and the Man.** We started this paper with the Lion and the Man and we close it with a simple interpretation of the results in this language. Assume we have a Riemannian manifold $M$ satisfying the condition in Theorem 12. Then, given a Brownian Lion running on $M$, Theorem 12 assures that there is a strategy for the Brownian Man which keeps him at the safe positive distance from the Lion for all times.

In addition, if the Ricci is non-negative, then the Brownian Man can choose a strategy which keeps him at fixed distance from the Brownian Lion. This must be particularly frustrating for the Lion especially if they start relatively close to each other.

Theorem 12 also shows that if the Ricci curvature is bounded below by a positive constant, then given a Brownian Man, the Brownian Lion has a strategy which will bring him arbitrarily close to its meal.

8.2. **Lower Bounds on Ricci Curvature.** As we pointed out in the introduction, [28, Corollary 1.4] shows that one can characterize the condition $Ric \ge k$ in terms of couplings. We can strengthen this a little bit in the following form.

**Corollary 19.** *Assume $M$ is a complete Riemannian manifold. Then the following two statements are equivalent.*

(1) *$Ric_x \ge k$ for all $x \in M$.*
(2) *For any point $z \in M$, there exist $r_z, \delta_z > 0$ such that for any $x, y \in B(z, r_z)$ we can find a Markovian coupling of Brownian motions $X_t, Y_t$ starting at $x, y$ with the property that*

$$d(X_t, Y_t) = e^{-kt/2} d(x, y) \text{ for } 0 \le t \le \delta_z \wedge \zeta_z$$

*where $\zeta_z$ is the first time either $X_t$ or $Y_t$ exit the ball $B(z, r_z)$.*

As a clarification, $X_t, Y_t$ need to be defined up to the exit time from the ball $B(z, r_z)$ or up to $\delta_z$, whichever comes up first.

*Proof.* The implication $1) \implies 2)$ follows from Theorem 16. The reverse implication we follow the same lines as in [28], particularly the implication x$\Longrightarrow$ and we will sketch only the main differences.

Instead of considering the heat kernel of the Laplacian on the manifold we consider the heat kernel $p_t(x, y)$ of half the Laplacian on $B(z, r_z)$ with the Dirichlet boundary conditions and its corresponding action $(p_t f)(x) = \int_{B(z,r_z)} p_t(x, y) f(y) dy$. Using this we can prove that condition 2) implies for any points $x, y \in B(z, r_z)$ and any compactly supported function $f$ on $B(z, r_z)$,

$$p_t f(x) - p_t f(y) = \mathbb{E}[f(X_{t\wedge\zeta_z}) - f(Y_{t\wedge\zeta_z})] \le |\nabla f|_{B(z,r_z)} d(x, y) \mathbb{E}[e^{-k(t\wedge\zeta_z)/2}]$$

from which one immediately gets by letting $y$ approach $x$ that

$$|\nabla p_t f(x)| \le |\nabla f|_{B(z,r_z)} \mathbb{E}[e^{-k(t\wedge\zeta_z)/2}].$$

Now, with very little changes in the argument of the implication v⟹i from [28], if $Ric_z(v,v) < k$ at some point $z$ for some $v$ we arrive at the following conclusion

$$k\mathbb{E}\left[1-\frac{t\wedge\zeta_z}{t}\right]\geq\epsilon+o(1)$$

for some $\epsilon > 0$. This certainly leads to a contradiction as we let $t \to 0$. □


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